International Union of Crystallography
Commission on Mathematical and Theoretical Crystallography

The enchanting crystallography of Moroccan ornaments

Satellite Conference of the ECM-24

Marrakech, Morocco, 20-22 August 2007
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The XXIV European Crystallographic Meeting was held from 22 to 27 August 2006 in Marrakech, Morocco.

The IUCr MaThCryst commission has organized a Satellite conference on Art and Crystallography devoted to the analysis of the Moroccan ornaments.

Venue

The satellite conference was held from 20 to 22 August, 2007. The first two days consisted of lectures delivered at the hotel Ryad Mogador Menara 5*, located in front of the Congress Palace where the main conference will be held. The last day has been devoted to an excursion to the Kasbah de Telouet, site renown for the richness of its ornaments.

Scientific director and guide of the excursion was Prof. Emil Makovicky, University of Copenhagen.

Program

Day 1

- 9:00-10:30: Symmetry in two dimensions (Takeo Matsumoto)
- 10:30-11:00 Coffee break
- 11:00-12:30: A crystallographer's view of Islamic ornaments: from Morocco to Iran and from plane groups to quasicrystals, part I (Emil Makovicky)
- 12:30-14:00 Lunch
- 14:00-15:30: A crystallographer's view of Islamic ornaments: from Morocco to Iran and from plane groups to quasicrystals, part II (Emil Makovicky)
- 15:30-16:00: Coffee break
- 16:30-18:30: Design and Construction of Islamic Geometric Patterns: historical Methodology and Contemporary Interpolation (Jay Bonner) cancelled

Day 2

- 9:00-10:30: Kaleidoscope of Moroccan ornamental art - a mathematician's view, part I: overview of the geometric decorative style in traditional Moroccan architecture, and some of its connections to contemporary mathematics. (Jean-Marc Castéra)
- 10:30-11:00 Coffee break
- 11:00-12:30: Kaleidoscope of Moroccan ornamental art - a mathematician's view, part II: workshop (Jean-Marc Castéra)
- 12:30-14:00 Lunch
- 14:00-15:30: Artist's and artisan's approach to, and practical realization of, Islamic ornaments (Jamal Benatia, Abdelaziz Jali, Abdelmalek Thalal)
- 15:30-16:00: Coffee break
- 16:30-18:00: Presentation of the trip to the Kasbah of Telouet (Emil Makovicky)

Day 3: Excursion to the Kasbah of Telouet: assisted practicals.
Symmetry in Two Dimensions

Takeo MATSUMOTO

Kanazawa University, Kanazawa, Japan
Symmetry in two dimensions

Takeo MATSUMOTO, Kanazawa Univ., Kanazawa, Japan

Private address: Tsuchisimizu 2-77, Kanazawa, Ishikawa, 920-0955, Japan.

The perception of symmetry in nature and in art has been appreciated since antiquity. The crystalline state of substance is characterized by a regular three dimensional repetition of atoms. This determines the macroscopic and microscopic characteristic features. A perfect crystal is considered to be constructed by the infinite regular repetition in space of identical structural units or building blocks.

The ornamental patterns show two or one dimensional symmetry. In this introductory lecture, we deal with symmetry as fundamental problem.

1. Symmetry and Crystal
   Crystal structure = lattice (translation, unit cell)
   + pattern (arrangement of atoms within the cell)

2. Symmetry operations and symmetry elements
   A symmetry operation of a given object is a motion (an isometric transformation) which maps this object onto itself.
   There are repetition operations, such as translation, rotation, reflection etc. All repetition operations are called “symmetry operations”.
   When a symmetry operation has a locus (a point, a line or a plane), that is unchanged, this locus is referred as the symmetric element.

   Seitz notation for transformation: \( \mathbf{x}' = \mathbf{R} \mathbf{x} + \mathbf{r} = \{ \mathbf{R}, \mathbf{r} \} \)
   \( \mathbf{X}' , \mathbf{x} : \) coordinates, \( \mathbf{R} : \) rotation part, 3x3 matrix, \( \mathbf{r} : \) a column matrix

   Rotation axes(proper rotation), minimum rotation(anti-clock) angle = \( 360^\circ/N \)
   \( N = 1, 2, 3, 4, 5, 6, 7, 8, \ldots \ldots \infty \)
   \( \text{determinant } \mathbf{R} = +1 \)

   Rotoinversion axes (improper rotation), rotation + inversion
   \( \overline{N} = 1, (\text{inversion}), 2, (\text{mirror}), 3(3+ \bar{1}), 4, 5, 6(3/m), 7, \ldots \ldots \)
   \( \text{determinant } \mathbf{R} = -1 \)
Crystallographic restriction due to lattice translations
R3: 1, 2, 3, 4, 6 and 1, 2, 3, 4, 6 in crystal point group,
R2: 1, 2, 3, 4, 6 and m.

3. Point groups
A point group is defined as a group of point symmetry operations whose operation at least one point unmoved.
There are 32 point groups (crystal classes) in space for crystal and 10 point groups in the plane. The former belongs to 7 crystal systems and the latter 4.
In each point group, the set of operations is a group from the mathematical point of view.

4. Space groups
The symmetry group of a three dimensional crystal pattern is called its space group. The operations of space groups include translations and rotations (proper and improper), and a more general type of operation resulting from the combination of a rotation and a translation.
New types of operation arise in certain space groups as follows.
Screw motions ……Screw axes,
Glide reflections…..Glide planes.
There are 14 essentially different lattices possible in R3 and 5 lattices in R2 and these are called the Bravais lattices.
Symbols …..Schoenflies, Hermann-Mauguin(full and short)
International Tables for Crystallography.

5. Crystallographic groups, \( G_t^n(l) \) (Neronova)
n : space dimension,
t : translation dimension,
l: multiple antisymmetry

\( l = 0 \) monochromatic
\( t = 0 \) point group; number: \( G_0^1 = 2, \ G_0^2 = 10, \ G_0^3 = 32, \ G_0^4 = 227. \)
\( t = n \) space group, \( G_1^1 = 2, \ G_2^2 = 17, \ G_3^3 = 230, \ G_4^4 = 4895. \)
\( l = 1 \) dichromatic \( G_{n-1}^n = G_n^n(l=1) \leftarrow \)
\( G_1^2 = 7 = G_1^1(l=1), \ G_2^3 = 80 = G_2^2(l=1), \ G_3^4 = 1651 = G_3^3(l=1). \)
6. Symmetry Groups $G_1^2$ of borders, Border patterns

These two-dimensional groups $G_1^2 \supseteq T_1 \ni t$ contain a translation in one, singular direction of two-dimensional space. 7 groups.

\[ p111, \ p1a1, \ p1m1, \ pm11, \ p112, \ pma2, \ pmm2. \]

7. Plane twice-periodic groups $G_2^2. \supseteq T_2 \ni t_1 , t_2$.

Group $T_2$ may be characterized by a basis – any pair of vectors forming the size of the primitive, unit parallelogram, and it consists of an infinite set of translations.

\[ t = ma + nb, \quad m,n = 0, \pm1, \pm2, \pm3, \ldots \ldots \]

Groups $G_2^2. \supseteq T_2$ may contain not only translations, but other operations. They are crystallographic, they can only have rotations 1.2.3.4.6.

Five different two-dimensional groups $T_2$ are called Bravais groups in $R_2$.

- oblique p,
- rectangular p and c,
- square p and hexagonal p.

<table>
<thead>
<tr>
<th>Oblique</th>
<th>$p1$</th>
<th>$p2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>pm, pg, cm</td>
<td>pmm2, pmg2, pgg2, cmm2</td>
</tr>
<tr>
<td>Square</td>
<td>p4, p4mm, p4gm</td>
<td>III</td>
</tr>
<tr>
<td>Hexagonal</td>
<td>p3, p3m1, p31m</td>
<td>IV</td>
</tr>
<tr>
<td></td>
<td>p6, p6mm</td>
<td>V</td>
</tr>
</tbody>
</table>

I (1360°), II (2,180°), III (4,90°), IV(3,120°), V(6,60°) smallest rotation -- George Baloglou

8. Anti-symmetry groups of one-sided bands, $G_1^2(l=1)$

Two colored border patterns.
Anti symmetry, color reversing operations.
Anti translation $t'$, anti mirror reflection $m'$, anti glide reflection $g'$ ($a'$, $b'$),
Anti identity $l'$, anti two-fold rotation (color reversing half turn) 2'
9. Layer group $G^3_2 = G^2_2(l=1)$ of three-dimensional two-periodic objects.

<table>
<thead>
<tr>
<th>Number of groups</th>
<th>31</th>
</tr>
</thead>
<tbody>
<tr>
<td>One colored</td>
<td>7</td>
</tr>
<tr>
<td>Neutral (gray)</td>
<td>7</td>
</tr>
<tr>
<td>Two-colored (black and white)</td>
<td></td>
</tr>
<tr>
<td>Without anti-translations</td>
<td>10</td>
</tr>
<tr>
<td>with anti-translations</td>
<td>7</td>
</tr>
</tbody>
</table>

Dichromatic plane groups, number of groups 80 (17 + 17 + 46)

Operations of motions: 1, 2, 3, 4, 6, m, g, t


10. Crystallographic orbit.

Crystallographic orbits are infinite sets of points due to the infinite number of translations in each space (or plane) group. Any one of its points may represents the whole crystallographic orbit.

References.

Crystallographic groups of four-dimensional space. New York: Wiley.
Space group, $S$

symmetry operation (rigid motion)

(an isometric transformation)

\[ S = (R, r) = R / r \]

\[
\begin{pmatrix}
X'_1 \\
X'_2 \\
X'_3
\end{pmatrix} =
\begin{pmatrix}
R_{11} & R_{12} & R_{13} \\
R_{21} & R_{22} & R_{23} \\
R_{31} & R_{32} & R_{33}
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
X_3
\end{pmatrix} +
\begin{pmatrix}
r_1 \\
r_2 \\
r_3
\end{pmatrix}
\]

\[ X' = R \cdot X + r \]

\[ (R_2, r_2) \cdot (R_1, r_1) = (R_2 R_1, R_2 r_1 + r_2) \]

$S$ (infinite group) \supset T (translation group, infinite group)

Factor group $S / T \cong$ Point group

isomorphic

1st kind operations, Proper

\[ \det R = +1 \]

Lattice translation $T$

Rotation $N$

Screw $P_d$

2nd kind operations, Improper

\[ \det R = -1 \]

Rotoinversion $\overline{N}$

Glide plane $a, b, c, n, d$

Rotation, Rotoinversion

<table>
<thead>
<tr>
<th>Trace R</th>
<th>Rotation det $R = +1$</th>
<th>Rotoinversion det $R = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>order</td>
<td>1 6 4 3 2</td>
<td>2 6 4 6 2</td>
</tr>
<tr>
<td>Type</td>
<td>1 6 4 3 2</td>
<td>1 6 4 3 m</td>
</tr>
</tbody>
</table>
\[
\begin{align*}
4_1 &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & 4_2^2 &= 2_1^2 &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & 4_3^3 &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & 4_4^4 &= 1_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\end{align*}
\]
Number of symmetry and multiple anti-symmetry groups.

$G_t^n(l)$.  $n$ : space dimension,  $t$ : periodicity,  $l$ : multiple-antisymmetry.

$l = 0$ : classical groups,  $l = 1$ : antisymmetry group  (modified, Neronova)

Takeo Matsumoto
Antisymmetry groups of one-sided bands

<table>
<thead>
<tr>
<th>One-colored</th>
<th>Neutral (gray)</th>
<th>Two-colored (black-white) without antitranslations</th>
<th>with antitranslations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p1 = (a) )</td>
<td>( p11' )</td>
<td>Orientation of axes</td>
<td>( p_a^{-1} )</td>
</tr>
<tr>
<td>( pm1l = (a) : m )</td>
<td>( pm11' )</td>
<td></td>
<td>( p_a^{-1} m_1 )</td>
</tr>
<tr>
<td>( p1m1 = (a) : m )</td>
<td>( p1m11' )</td>
<td></td>
<td>( p_a^{-1} a_1 )</td>
</tr>
<tr>
<td>( p1a1 = (a) : \tilde{a} )</td>
<td>( p1a11' )</td>
<td></td>
<td>( p_a : m_2 )</td>
</tr>
<tr>
<td>( pmm2 = (a) : 2m )</td>
<td>( pmm21' )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( pma2 = (a) : 2 \tilde{a} )</td>
<td>( pma21' )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( p112 = (a) : 2 )</td>
<td>( p1121' )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( p_a : 112 )</td>
</tr>
</tbody>
</table>

Fig. 208. International notation and geometric realizations of the antisymmetry groups of one-sided bands composed of asymmetrical triangles.

Shubnikov, Koptsik, «Symmetry in Science and Art»
Fig. 2.56. (a) Graphical representation of 17 plane groups $G_7$ [2.16]

Vainshtein, "Modern Crystallography"
**p4gm**

Square 4 mm

**Origin at 4**

<table>
<thead>
<tr>
<th>Number of positions, typical sequence, and point symmetry</th>
<th>Co-ordinates of equivalent positions</th>
<th>Conditions limiting possible reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 d 1</td>
<td>$x,y; y,x; \pm x, \mp y; \pm y, \mp x$</td>
<td>General: no conditions</td>
</tr>
<tr>
<td></td>
<td>$x/2, y/2; y/2, x/2; \pm x/2, \mp y/2; \pm y/2, \mp x/2$</td>
<td>A: no conditions; B: $2\pi$ (for $x=2\pi$)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>B: no conditions</td>
</tr>
<tr>
<td>4 e m</td>
<td>$x/4 \pm x; x/4 \pm -x; y/4 \pm y; y/4 \pm -y$</td>
<td>Special: as above, plus no extra conditions</td>
</tr>
<tr>
<td>2 b mm</td>
<td>$x/3; y/3; 0,1$</td>
<td>A: $x=\pm 2\pi$</td>
</tr>
<tr>
<td>2 a 4</td>
<td>$0,0; \pm x, \pm y$</td>
<td>A: $x=\pm 2\pi$</td>
</tr>
</tbody>
</table>

**Crystallographic orbit**

$8 \cdot d \cdot 1 = \{x,y\} 1.p4gm[a,b] \rightarrow \{x,x\} 1.m1.p4mm[a',b']$

$\rightarrow \{x,0\} 1.m.p4mm[a',b'] \rightarrow \{1/4,1/4\} L.t[a'/2,b'/2]$  

$4 \cdot c \cdot m = \{x,1/2+x\} m.p4gm[a,b] \rightarrow \{1/4,1/4\} L.t[a/2,b/2]$  

$2 \cdot b \cdot m = \{1/2,0\} L.t[a',b']$  

$2 \cdot a \cdot m = \{0,0\} L.t[a',b']$  

$L.t = 4mm.p4mm$, square lattice

$a'=(a-b)/2$, $b'=(a+b)/2$, $a''=(a-b)/4$, $b''=(a+b)/4$

**Dichromatic group**

$pc'4gm$, $p4g'm$, $p4g'm'$, $p4g'm'$. 
| No. 7 | p 2 mg | m m | Rectangular |

### Origin at 2

<table>
<thead>
<tr>
<th>Number of positions, Wyckoff notation, and point symmetry</th>
<th>Co-ordinates of equivalent positions</th>
<th>Exception</th>
<th>Orbit types</th>
<th>Conditions limiting possible reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 d 1 x, y; x, y; ( \pm x, y; \pm x, y )</td>
<td>( d_1, d_2, d_2', d_2'' )</td>
<td>( d_4, d_4', d_4'' )</td>
<td>1.p2mg</td>
<td>General: ( hk ): No conditions ( h0 ): ( h=2n )</td>
</tr>
<tr>
<td>2 c m 1,x; 1,x</td>
<td>( c_1, c_1', c_2, c_2' )</td>
<td>( { L_0 } )</td>
<td>1m.l.p2mg (( = m.p2mg ))</td>
<td>Special: as above, plus no extra conditions ( hk ): ( h=2n )</td>
</tr>
<tr>
<td>2 b 2 0,1; 1,1</td>
<td>[ ]</td>
<td>( { L_0 } )</td>
<td>h 2n</td>
<td></td>
</tr>
<tr>
<td>2 a 2 0,0; ( \pm y )</td>
<td>[ ]</td>
<td>( { L_0 } )</td>
<td>h 2n</td>
<td></td>
</tr>
</tbody>
</table>

| 4 d_1 * 0,y; 0,y; 1/2,y; 1/2,y | d_4 | 1m.l.p2mg \( \{ \frac{a}{2},b \} \) | \( hk \): \( h=2n \) |
| 4 d_2 ** x,0; \( \pm x,0 \); 1/2,\( \pm x,0 \); 1/2,-x,0 | d_5 | 1m.p2mm \( \{ \frac{b}{2},b \} \) | \( |k| : h=2n \) |
| 4 d_2 ** x,1/2; \( \pm x,1/2 \); 1/2,\( \pm x,1/2 \); 1/2,-x,1/2 | d_5 | \( \{ L_0 \} \) | \( hk \): \( h+k=2n \) |
| 4 d_3 *** x,1/4; \( \pm x,3/4 \); 1/2,\( \pm x,3/4 \); 1/2,-x,1/2 | d_4 | 1m.c2mm | \( hk \): \( h+k=2n \) |
| 4 d_4 **** 0,1/4; 0,3/4; 1/2,3/4; 1/2,1/4 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h+k=2n \) |
| 4 d_5 ***** 1/8,0; 3/8,0; 5/8,0; 7/8,0 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h=4n \) |
| 4 d_5 ***** 1/8,1/2; 3/8,1/2; 5/8,1/2; 7/8,1/2 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h=2n \) |
| 2 c_1 * 1/4,0; 3/4,0 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h+k=2n \) |
| 2 c_1 * 1/4,1/2; 3/4,1/2 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h=2n \) |
| 2 c_2 ** 1/4,1/4; 3/4,3/4 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h+k=2n \) |
| 2 c_2 ** 1/4,3/4; 3/4,1/4 | \( L_0 \{ \frac{a}{2},b \} \) | \( hk \): \( h+k=2n \) |

\* \( m/2,1 \); ** \( 1,g/2 \); *** \( 1,g/2 \); **** \( m/2,g/2 \); ***** \( (m & 2)/2,g \)
\* \( m,g \); ** \( m,g/2 \).
### pmg

**No. 7  p2mg  mm  Rectangular**

**Origin at 2**

<table>
<thead>
<tr>
<th>Number of positions, Wyckoff notation, and point symmetry</th>
<th>Co-ordinates of equivalent positions</th>
<th>Conditions limiting possible reflections</th>
</tr>
</thead>
<tbody>
<tr>
<td>4  d  1  x,y; 2x,y; 1+x,y; 1-x,y.</td>
<td></td>
<td>General:</td>
</tr>
<tr>
<td>2  c  m  (\frac{1}{2}, y); (\frac{3}{2}, y).</td>
<td></td>
<td>(hk): No conditions</td>
</tr>
<tr>
<td>2  b  2  0,(\frac{1}{2}); (\frac{1}{2}, \frac{1}{2}).</td>
<td></td>
<td>(h0): (h=2n)</td>
</tr>
<tr>
<td>2  a  2  0,0; (\frac{1}{2}, 0).</td>
<td></td>
<td>Special:</td>
</tr>
<tr>
<td>4  d  1  (x, y)</td>
<td></td>
<td>as above, plus</td>
</tr>
<tr>
<td></td>
<td></td>
<td>no extra conditions</td>
</tr>
</tbody>
</table>

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]

**2 c m \(\frac{1}{2}, y\); \(\frac{3}{2}, y\).**

**2 b 2 0,\(\frac{1}{2}\); \(\frac{1}{2}, \frac{1}{2}\).**

**2 a 2 0,0; \(\frac{1}{2}, 0\).**

**4 d 1 \(x, y\)**

\[1\text{. p2mg}(a, b)\]
Table 7-b Plane Patterns — Fundamental Types and their Symmetries (Two-color)

36 Pbc2
37 Pnc2
38 Pnm2
39 Bbm2
40 Bba2
41 Pbn2
42 Pmn2
43 Ccc2
44 Ccm2
45 Imm2
46 Iba2
47 Ibm2
48 P4
49 P
50 P4gm
51 P4m'm'
52 P4mm
53 P4mm
54 P4mm
55 P4mm
56 P4mm
57 P4mm
58 P3c1
59 P3c1
60 P3m1
61 P6
62 P6
63 P6
64 P6
65 P6
66 P6
67 P6
68 P6
69 P6
70 P6
71 P6
72 P6
73 P6
74 P6
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98 P6
99 P6
100 P6
101 P6
102 P6
103 P6
104 P6
105 P6
106 P6
107 P6
108 P6
109 P6
110 P6
111 P6
112 P6
113 P6
114 P6
115 P6
116 P6
117 P6
118 P6
119 P6
120 P6
121 P6
122 P6
123 P6

S. Kamagai, Y. Sawada, 'Ornamental Patterns and Symmetry'
Packings of circles and ellipses with 6 contacting neighbours. Nowacki (1948)

Packings of ellipses with 6 contacting neighbours. Grünbaum and Shephad (1987)

Three p2gg packings of ellipses with six contacting neighbours.

We can distinguish them by symmetry and mutual ratios of cell dimensions, in the case of the closest packing of circles, namely k=1 (the densest packing, p6mm).
S. Kumagai, Y. Sawada, 「Ornamental Patterns and Symmetry」
Islamic Ornaments: Fundamental Language of Symmetry

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ISLAMIC ORNAMENTS: FUNDAMENTAL LANGUAGE OF SYMMETRY
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INTRODUCTION
SYMMETRY OPERATIONS AND UNIT MESH
FIVE SYSTEMS
PLANE GROUPS OF SYMMETRY
DICHROIC PLANE GROUPS
POLychROMATIC GROUPS OF SYMMETRY
LAYER GROUPS OF SYMMETRY
USAGE
FRIEZE GROUPS AND ROD GROUPS
SIMILARITY AND HOMOTHETY
POINT GROUPS OF SYMMETRY IN PLANE AND SPACE
MORE ON USAGE: PATTERN COMPLEXITY, SYMMETRY CONTENTS, RELATIVITY OF INTERPRETATION
QUASICRYSTALLINE PATTERNS
STRUCTURAL CLASSIFICATIONS
MATERIAL AND STYLE
KASBAH DE TEOUET

INTRODUCTION
Ornamental art is like any creation and even life itself: freedom of choice within rules. And these rules are symmetry in its many forms and different meanings: even asymmetry is a form of symmetry. This contribution enumerates fundamental types of symmetry found in two-dimensional Islamic art and describes their basis and applications. For sure, Islamic artists had notions of symmetry at least partly different from our ‘modern’ ones, refined on literally thousands of crystal structure descriptions and hundreds of mathematical treatises since the time of the Erlangen programme of Klein in 1870’s. However, this does not prevent us from studying and defining the
results of their endeavours: we stand in admiration in front of the Kharraqan towers in Iran, or Alhambra in Spain but we stand in awe in front of their quasicrystalline ornaments in Maragha or Sevilla- they beat us by 800, resp. 650 years!

**Symmetry operations and unit mesh**

The majority of Islamic ornamental patterns are two-dimensional periodic patterns by their nature. In such patterns a single motif (e.g., an individual lozenge, rosette or star) or several distinct motifs repeat regularly and infinitely. All repetitions of a given motif are related to each other by one or several exactly defined ways, called operations of symmetry. All symmetry equivalent motifs have the same shape and size and the same surroundings from the point of view of symmetry and geometry. The symmetry-equivalent motifs can be related to one another by translation (pure displacement without a change in orientation)(Fig. 1), reflection (on a reflection axis in the plane of the pattern or on the reflection (= mirror) plane m perpendicular to the plane of the pattern, depending on the nomenclature used by the particular author, Figs. 1,4) or rotation (by an angle of n/360° around an n-fold rotation axis perpendicular to the plane of the pattern; the values of n permitted in regularly repeating patterns are 2, 3, 4 and 6, Fig. 2). A combination of reflection with translation (by \( \frac{1}{2} \) of the full repetition period of the periodic pattern) gives glide-reflection planes (axes) g (Fig. 4).

From the infinite number of translation vectors between the periodically displaced copies of the same motif, that can be found in a two-dimensionally periodic pattern, we usually select the two shortest ones. These comprise an angle of 90°, 120° or quite a general obtuse angle and they are equal or not equal in length and kind (e.g., Figs. 1-6). The two vectors define a parallelogram, by the translation (displacement) of which the entire pattern can be reconstructed. This parallelogram is called a unit mesh (unit cell). The unit mesh is primitive when the translation-equivalent copies of the motif are only in its corners or it is centred when an additional copy lies in its centre (Fig. 1).

**Five systems**

Only five basic shapes of unit mesh exist in planar patterns. All the patterns with a
given type of unit mesh belong to the same system. These systems are as follows (a and b are the selected vectors, γ is their angle):

(1) Oblique \(a \neq b, \gamma \neq 90^\circ\); symmetry represents pure translations or translations and 2-fold axes.

(2) Rectangular \(a \neq b, \gamma = 90^\circ\). Symmetry elements present: reflection planes and glide-reflection planes that can be combined with each other and with 2-fold axes.

(3) Diamond (rhombic, centred rectangular) system which can either be described by a diamond-shaped mesh \(a = b, 120^\circ \neq \gamma \neq 90^\circ\) or by a rectangular mesh with an additional translation-equivalent point in its centre (so-called centred mesh \(a \neq b, \gamma = 90^\circ\), Fig.1). Relevant symmetry represents a combination of reflection and glide-reflection planes with or without 2-fold axes.

(4) Square \(a = b, \gamma = 90^\circ\); 4-fold axes combined with 2-fold axes; with or without reflection and glide-reflection planes.

(5) Hexagonal (equilateral-triangular) \(a = b, \gamma = 90^\circ\), three-fold axes with or without reflection and glide-reflection planes or 6-fold axes combined with 3- and 2-fold axes, with or without these planes of symmetry.

All of these systems are relevant for the study of Islamic ornaments of different epochs; especially frequent are the fourth and fifth one (Makovicky & Makovicky 1977, Abas & Salman 1995: 138). Many of the Islamic patterns are metrically hexagonal, square or rectangular in spite of low symmetry (i.e., their symmetry might require only the oblique system).

**Plane groups of symmetry**

The operations of symmetry in plane form several contradiction-free combinations, the so-called plane (or two-dimensional) symmetry groups. Within these contradiction-free combinations of symmetry operations, any operation can be replaced by a combination of two or more other symmetry operations which are present in the given symmetry group. When we combine two or more symmetry operations, the result does not depend on the sequence of operations we have chosen. Finally, any operation in the group has its inverse - a combination of a symmetry operation and its inverse results in the identity of the transformed motif with its original. Only seventeen distinct plane
groups of symmetry can be found in all two-dimensional patterns, all of which have been encountered in the Islamic ornaments even if not at the same time and place. As it transpires from the above listing of two-dimensional systems, the presence of certain rotation axes and, for some cases, also the presence/absence of reflection planes in a plane group determines the geometry of unit mesh and the system to which the plane group belongs. In this way, symmetry and geometry of plane patterns are inextricably interconnected.

Books on symmetry written by mathematicians often start with abstract mathematical groups, proceed then to derive the two-dimensional groups of symmetry and not always get down to the problems of their practical application. An exception is the book by Abas & Salman (1995) in which a profusion of examples with the basic symmetrological information attached are given. A non-mathematical approach to plane groups is taken by Washburn & Crowe (1998: 58) in a book intended to teach anthropologists 'the symmetries of culture'.

Crystallographers are perhaps the most frequent practical users of symmetry groups. The approaches and notions used in the present review come partly from this experience. The chapters on symmetry in crystallographic textbooks range from highly mathematical (but always practical) to almost purely geometric; the reader can find an understandable/complete approach to plane groups in Klein (2002).

The crystallographic notation of plane groups used here consists of four-place symbols. The first letter p or c denotes primitive and centred mesh, respectively; an integer n denotes the highest order of rotation (two-fold, three-fold, four-fold and six-fold rotation is allowed for periodically repeating patterns whereas five-fold and eight-fold axes, as an example, are not allowed in them). In the symbols with maximally two-fold axes present, the next letter denotes a mirror plane m or a glide-reflection plane g perpendicular to the a axis (which points downwards in each figure); and the following letter indicates such planes perpendicular to the b axis (which points to the right). The trivial factors (the directions to which no perpendicular symmetry planes exist), however, are omitted in the abbreviated symbols commonly used. They would have been denoted by "1" on the relevant position of a full symbol. Other rules of notation are
valid for the square and hexagonal systems, with 4-fold, 3-fold and 6-fold axes. Here, the first letter symbol denotes planes perpendicular to all the mutually equivalent a axes and the last letter symbol those perpendicular to the diagonals between adjacent a axes. These diagonals comprise 45° with the axes in the tetragonal (square) system and 30° in the hexagonal (triangular) system.

Examples of plane groups of the first category are \(p1\), \(p2\), \(p2gg\), \(c2mm\), those of the second category are \(p3\), \(p31m\), \(p4gm\), \(p6\), \(p6mm\).

Application of plane groups of symmetry does not require their (re)derivation, but it depends critically on their right interpretation. In this connection, division of elements of the motif into those placed in general positions (not situated on any element of symmetry) and in special positions (situated on an element of symmetry or in an intersection of several elements of symmetry) is of the most fundamental importance although it is rarely mentioned outside the field of crystallography. The position in which it is placed determines both the number of times it occurs in a unit mesh (multiplicity) and the symmetry of the motif. An element (i.e., motif) in the special position ought to have symmetry at least equal to the symmetry elements on which it is positioned. If its symmetry is lower, it reduces the overall symmetry of the pattern creating a less symmetric plane group; if it is higher, sometimes it may result in stacking errors of the design. An element in general position is asymmetric by virtue of its position. If it has any other symmetry, it is in excess of the local requirements. As an example, the elements of, and the general and special positions in, several plane groups are illustrated in Figs. 1-2. For example, the \(cmm\) group has a centred unit mesh and two sets of mirror planes, respectively perpendicular to the a and b axes. As it can be seen in Fig. 1, these "principal symmetry elements", which form part of the plane group symbol, are interleaved by glide-reflection planes. Two-fold axes occur on intersections of all symmetry planes of the same kind. Motifs can be placed at different choices of the general position, on mirror planes, on two-fold-axes or on \(2mm\) intersections (Fig. 1).

**DICHROIC PLANE GROUPS**

A simple extension of groups of symmetry can be achieved when a change of colour
of the motif acted upon is assigned to each step of a particular operation of symmetry. A change of colour from "white" to "black" (and back in the next step) can be assigned to translations, centration, 2-, 4- or 6-fold rotation as well as to the reflection and glide-reflection planes (Fig. 4). Such operations are called operations of antisymmetry and the symmetry groups are called dichroic or black-and-white, or groups of antisymmetry. In the case of 2-dimensional periodic patterns in plane, 46 plane groups can be discerned when the above 17 uncoloured plane groups are not counted. In the symbols, the colour changing symmetry operators are primed, the colour-unchanging ones are left unprimed (e.g. \( p2' \), \( cm'm \), or, finally, \( pb' mm \), when the \( b \)-translation is an operation of antisymmetry). One plane group will yield several dichroic groups on colouring, by application of antisymmetry, to certain symmetry operators. Thus, a dichroic group can be described by the symbol \( pC'mm2 \) when \( C' \) is the colour-changing operation of mesh centration. On the other side of the spectrum is a \( cm'm' \) pattern in which both systems of mirror planes and the associated glide-reflection planes change colour on reflection whereas the centration vector and all two-fold rotation axes preserve the colour of elements they act upon (Fig. 6).

The two simplest dichroic colourings of the \( p2 \) pattern yield following results: In the first case, all two-fold axes become colour-changing, resulting in symmetry \( p2' \), whereas in the second case only a half of their set will represent colour-changing operators. The latter case can be conveniently described as a doubling of one of the unit-mesh edges by a colour-changing translation, in our case as \( pa'2 \).

**POLYCHROMATIC GROUPS OF SYMMETRY**

A rectangular \( pmm \) example with dichroic translation \( b' \) or \( a' \) turns into \( pa'mm \) or \( pb'mm \) (Fig. 5). This leads towards a generalization of this process for more than two colours and to the simplest type of coloured groups of symmetry in which colours of given elements of a pattern change periodically along one translation direction. Along this colour-modulation direction the fundamental translation of the pattern will change to its \( n \)-tuple for the sequence of \( n \) colours involved (Figs. 11,12).

Thus, trichroic modulation of the \( pmm \) pattern will result in a trichroic plane symmetry
group with a tripled $b$-translation. If we study the chromaticity (colour action) of mirror planes perpendicular to the colour-modulated $b$ direction as well as that of two-fold axes, we shall find that it is only partial for this case. These symmetry elements alternate only two colours, leaving the third one unchanged. The resulting plane group symbol is correspondingly complicated, $p_{b(3)}mm^{(2,1)}2^{(2,1)}$. Superscripts in round brackets before the comma indicate number of permutated colours and after the comma, number of unchanged colours. When ignoring the symmetry of the underlying pattern, this plane group can be simplified to $p_{b(3)}m1$; in this case the colour scheme employed is presented as simple vertical colour stripes overlying the correspondingly coloured diamonds.

There are also other types of polychromatic groups, although many cultures started, and often stayed with, the above colour-modulated cases (Makovicky 1986). They even appear to be a starting point of modern painters, such as Escher and Hinterreiter. All symmetry elements of plane groups can become operators of coloured symmetry, with the colours of the pattern changing in a predesignated sequence when all the steps of the symmetry operation are performed. By different choices of colour sequences for these operators, different polychromatic group can be derived from the same plane group. Superscripts in parentheses indicate number of colours permutated by the given symmetry operator acting upon an object in general position of the group. Thus, $m^{(2)}$ is a two-coloured reflection plane, $a^{(3)}$ a three-coloured and tripled translation, and $6^{(6)}$, $6^{(3)}$ and $6^{(2)}$ a six-fold axis with cyclic colour sequences ABCDEF, ABCABC, and ABABAB, respectively. The total number of colours (of a general position) in a group (its colour index) may be equal to, or exceed, the highest colour index of a single operation. It is important to realize that polygons situated on special positions will show lesser colour variation, their colour being a combination of two, three, four or more colours of general positions which coalesced into this special position. For example, a disc positioned on $6^{(6)}$ will be coloured by a combination of 6 different colours, maintaining the same colour for all partial 60° rotations although the adjacent general position shows a sequence of six colours.

In several cultures, among which the Islamic one is prominent, a repeated dichroic division of elements into a white and a coloured subset was common. Usually the
‘white’ elements are left intact and the ‘black’ ones are again divided into two differently coloured subsets, which can be called the ‘white’ and ‘black’ subsets of the second stage (Fig. 14). As this operation may be repeated several more times, these patterns were defined as sequentially dichroic by Makovicky & Fenoll (1999); examples from different cultures (especially Islamic and Old Egyptian) are given in Makovicky (1986).

The use and profusion of these examples, together with the relative paucity of the above multicoloured ‘truly’ polychromatic groups with the appropriate polychromatic symmetry operators has been noted by Makovicky (1986), who postulated that the latter are, in essence, limited to modern times, and mastered especially thanks to the efforts of Escher and Hinterreiter (Makovicky 1979).

**LAYER GROUPS OF SYMMETRY**

Although truly two-sided window trellis are not common, Islamic ornamental art produced a multitude of interlaced ornaments, which by their nature are two-sided ornaments, with one side oriented toward the viewer and the other side into the background (Figs. 15-17).

The plane ornaments which have been treated amply in the previous text, are 2D patterns in a 2D space (i.e., in a plane), i.e. from the point view of symmetry do not have sides. The two-sided ornaments are layer patterns, i.e., 2D patterns in 3D space, and their symmetry is described by ‘layer groups of symmetry’. Thus, they have two sides, which in the layer groups of symmetry may be related via reflection, glide reflection, inversion, or by two-fold rotation about binary axes lying in the plane of the layer. These are so called non-polar groups. The cases in which these opposing sides remain unrelated are polar groups of symmetry. There are 80 layer groups of symmetry, distributed over the same systems as the plane groups.

There is a complete formal correspondence between dichroic (black-and-white) groups of symmetry and layer groups of symmetry. The ‘black’ and ‘white’ elements
of the former correspond to the ‘upper’ and ‘lower’ elements of the latter. Thus, the
dichroic plane group $\text{p}6\text{m}'\text{m}'$, derived by colouring from the plane group $\text{p}6\text{mm}$,
corresponds to the layer group $\text{p}6\text{2}2$ with two-fold axes in the directions of $a$ axes and
their diagonals, respectively (Fig. 17). Reflection on the mirror plane situated in, and
parallel to the layer corresponds to ‘grey groups’ of dichroic classification, with both
colours in each element; the polar groups correspond to plane groups.

**USAGE**

The advantages of using plane groups of symmetry for analysis of larger bodies of
ornaments are undisputable today (e.g., Makovicky & Makovicky 1977, Makovicky
symmetry statistics from different sets/objects or cultures, Makovicky and Fenoll
(1997) devised a rosette diagram in which plane groups (and systems) are distributed
according to the unit rotations present in them, from 1-fold to 6-fold rotation axes (Fig.
23-24). In each group, a progression from the groups with purely rotational set of
operators to those with maximum of reflection planes occurs. Frequency of cases
observed is plotted radially; places for quasicrystalline aggregates have been reserved
as well.

Description of coloured ornaments by means of dichroic or polychromatic groups of
symmetry is less widespread but equally effective (e.g., Makovicky 1986, Washburn
& Crowe 1998, Makovicky & Fenoll 1999). On the one hand it gives a shorthand
notation for the colouring schemes applied, on the other it allows to pick out colouring
sequences built on the same principle notwithstanding the colour combinations or
element shapes used. Dichroic or polychromatic groups of symmetry are the best tool
for further analysis of distinct sequences based on the same underlying pattern.

If we ignore intertwining of linear elements in a two-sided (layer) pattern, we can
tentatively reduce a layer group to the corresponding plane group (see the above
example of $\text{p}6\text{2}2$), especially when constructing statistics of symmetry groups for an
architectural object/period in which both plane- and layer groups were used. The same
is true for the patterns with dichroic and polychromatic groups of symmetry: for the
purpose of the first-stage statistics, their colouring can be ignored and patterns can be
grouped according to the underlying, uncoloured symmetry. If desired, the true plane
group patterns and the ‘simplified’ patterns can be distinguished by different colour of
radial vectors in Fig. 23. Some authors, in their chase for ‘a complete set of plane
groups present’ ignored the difference between uncoloured and coloured plane groups
of symmetry and took some dichroic groups as plane groups in which ‘black’ and
‘white’ elements are non-equivalent. This is a fundamentally wrong approach,
however, and it distorts the real situation.

**FRIEZE GROUPS AND ROD GROUPS**

One-dimensional periodic patterns are in common use as borders of panels, tympana,
doors, and windows or as dividers between different portions of a building. All the
geometrical variants notwithstanding, there are only seven frieze-groups of symmetry
they must obey. The lowest one, \( p_1 \), consists of pure repetitions of the motif by
translation whereas the highest one, \( p_{2mm} \), has a mirror plane parallel to the frieze,
regularly repeating mirror planes perpendicular to it, and two-fold axes in their
intersections. If glide-planes are present, as in \( p_{11g} \) and \( p_{2mg} \), they are parallel to the
frieze. A scheme for statistics is proposed in Fig. 25; it separates these groups into
those without two-fold axes and those with them – based on visual importance of this
symmetry element.

There are two types of rod groups in Islamic (and other architectural) art:
interlaced/intertwined, two-sided ornaments applied to flat background and all-round
ornamentation of columns and pillars. There are 22 rod groups for the first type of
ornaments, including the above seven groups for one-sided, plane ornaments. Groups
for all-round ornamentation are not limited to the crystallographic set of rotation axes
parallel to the column axis – they can also be 5- or 7-, 8-fold, etc.

With the multiplicity of this axis also increases the ambiguity of interpretation – it may
be more relevant to consider them as plane/layer group patterns wrapped around the
column shaft; we often do just that.
SIMILARITY AND HOMOTHETY

Similarity operations preserve the shape of the motif unchanged and change only its size. Homothety is the basic operation, in which equivalent points of similar objects are connected by rays emanating from a singular point and their dimensions and spacing are related by the equation

\[ \mathbf{A'B'} = k \mathbf{AB}, \]

where \( k \) is a similarity coefficient. Homothety can be combined with rotation or reflection. Thus, three kinds of similarity operations exist in plane: homothety, spiral rotation combining rotation and (centrifugal) homothety, and homothetic reflection, which is a similarity analogue of a glide-reflection plane. For two-sided patterns, a similarity analogue of an ordinary two-fold axis exists. Rotation is not limited to crystallographic angles.

Groups of similarity and rod groups on columns have a lot in common: the rotation axis parallel to the column becomes the central rotation axis of cartwheel patterns with similarity, and the translations along the column shaft become radial operations of homothety. Spiral winding around the column becomes spiral rotation as defined above. There is 1:1 correspondence between these two types of groups.

POINT GROUPS OF SYMMETRY IN PLANE AND SPACE

Circular, disc-like, square- or lozenge-shaped and rectangular ornaments of limited extent display point-group symmetry, in which all symmetry elements (valid for entire ornament, not only for its portion) intersect in the center point. In direct analogy to two-dimensional ornaments discussed previously, they can be in plane without any dimension in a direction perpendicular to the plane [they do not have two sides] (they can be uncoloured, black-and-white, and polychromatic when applicable) or they can be situated in 3D space, i.e., have ‘upper’ and ‘lower’ surface (and display intertwining/interlacing).

Point groups in 3D space are those known from crystal morphology. However, the crystallographic limitation (from the underlying periodic structure) on the rotation axes
(only 2-, 3-, 4-, and 6-fold axes allowed) and resulting point groups is absent here: for example, 5-, 8-, and 10-fold axes were popular in Islamic art; icosahedral system occurs as well.

Therefore, each of the point groups of morphological crystallography is a member, 'prototype' of an infinite family of point groups with 'crystallographically forbidden' rotation axes. Except for windows, groups with a mirror reflection parallel to the drawing plane are not of interest – interlacing generates inversion and/or two-fold axes in the drawing plane.

Point groups in plane (2D space) are correspondingly simpler than those in space. Point groups without reflections form a series 2, 3, 4, 5, 6, 7, 8, ...; those with reflections \( m \), \( 3m \), \( 5m \), \( 7m \), and \( 2mm, 4mm, 6mm, 8mm \), respectively. Possibilities of dichroic and, where applicable, also polychromatic colouring are obvious, e.g., \( 3^{(3)} \), \( 2m'm' \), \( 6'm'm' \), \( 6^{(6)}m'm' \), and \( 6^{(6)}m'm \).

Besides obvious applications of point-group symmetry, there exist a number of boundary cases, discussed in the next section.

**MORE ON USAGE: PATTERN COMPLEXITY, SYMMETRY CONTENTS, RELATIVITY OF INTERPRETATION**

Practical experience reveals very soon that lumping of all pattern types into a single diagram of plane group symmetries may obscure their real distribution. A much more revealing picture may be obtained when symmetries of the following categories are studied independently:

1. simple patterns with one motif present (one Wyckoff position occupied)
2. patterns of intermediate complexity with 2-3 motifs (occupied Wyckoff positions) present
3. complex patterns consisting of many motifs (tile types) and usually with a multiple occupation of certain Wyckoff positions.
4. (4)

This classification may be straightforward in the case of shaped tiles, but choices have
to be made in the case of motifs (visually prominent elements) without clear boundaries or the line motifs – should we choose intersections of lines or the fields enclosed by them? More on pattern complexity and its various aspects you can find in Makovicky and Fenoll (1997).

‘Symmetry contents of a symmetry group’ is an unsolved, controversial concept. In crystal structures, elements (=polyhedra) positioned on certain Wyckoff positions have their point-group symmetry reduced automatically to that of the position. It is not so in the man-made patterns, especially in the numerous Islamic ornaments. The spectator alternatively perceives the plane group symmetry of the ornament and the point-group symmetries of individual tiles or of the visually prominent aggregates of tiles, e.g. rosettes or stars. Each element is in a Wyckoff position, which has its plane-group multiplicity (number of times it occurs in a unit mesh) and the multiplicity of the point-group symmetry of this position. A product of these two is constant for the given plane group.

However, in Islamic patterns often the point-group symmetry – and the corresponding multiplicity - of the element in a Wyckoff position exceeds the ‘natural’ site symmetry of that Wyckoff position, e.g. an octagon positioned in the position mm2. Then the products of the two multiplicities exceed the value appropriate for the plane group; as a consequence, an average of products calculated for all elements of a pattern is a measure of its visual symmetry content. A ratio of this value, divided by the same product for unmodified plane group is then a visual symmetry index.

Except for the simplest patterns, interpretations of pattern complexity and symmetry content depend somewhat on the choice of elements to be treated by the observer. Although not perceived as such, this situation occurs in structural crystallography as well – it is us who select and define polyhedra to be treated. In the case of art we should try to understand what were the principal elements in the eyes of pattern creators, what were the eye-attracting aggregates and what may just have been the ‘cement’ between them. This may lead to different degrees of depth in the treatment of complex patterns, adjusted to the purpose and generality of the study.
A very important practical problem is the interpretation of frieze and two-sided rod patterns. They can be divided into three groups:

(a) truly one-dimensional patterns
(b) clear cut-outs of two-dimensional patterns (mostly with unambiguous propagation scheme)
(c) suspected cut-outs of two-dimensional patterns with boundaries modified in a way which prevents their propagation.

For the case sub (b) and eventually (c), the plane group of the two-dimensional pattern is to be defined and the number of periods across the frieze given. Examples are in Makovicky and Fenoll (1997) and practical examples in Schneider (1980).

Application of point-group symmetry to real ornaments may meet with interpretation problems: what is the extent (size) at which a complicated ‘carpet’ or ‘Pretzel’ ornament converts from a 0D object with point-group symmetry into a cut-out of a 2D ornament with plane- or layer-group symmetry, or into such a fragment of a frieze-(rod-)like ornament? What is the area ratio of inner (1D or 2D-like) portions to the modified boundary portions at which such a switch in interpretation becomes justified?

How to treat an ornament that is an obvious result of (a) twin operation(s) applied to a 2D pattern? This is a frequently overlooked case.

**QUASICRYSTALLINE PATTERNS**

The review of plane group symmetry given above reveals parallelograms, rectangles, squares, triangles and hexagons as the unit mesh, which can fill the plane. Pentagons, octagons and dodecagons are missing from this list, together with heptagons, etc. In 1974, Penrose proved that an agglomerate of pentagons and intervening 36° lozenges, the latter occurring isolated or clustered into pentagonal stars and half-stars, will yield aperiodic tiling when furnished with appropriate edge-matching conditions (markings). The number of original ‘prototiles’ has been reduced to two in the version of Penrose tiling which consists of ‘darts’ and ‘kites’ (both with a 72° vertex)(Fig. 26) and in the version composed of two Penrose rhombs, with 36° and 72° vertices, respectively. Both versions are furnished with appropriate edge- and vertex matching conditions, which, unfortunately, are missing in most popular articles. A very important property of
these tilings is self-similarity: several tiles are combined into larger tiles of the same kinds and disposition as the original ones, and so on, without end (Fig. 42). Only aperiodic tilings have this property.

In 1977, Ammann discovered octagonal aperiodic tilings: the ‘A4’ set that consists of two different indented tiles with right angles only and the A5 set of 45° lozenges and squares. Again, both have vertex and edge markings in order to enforce aperiodicity. Details of this tiling and the dodecagonal aperiodic tiling were worked out by Socolar (1989)(Figs 34-36).

Ammann discovered that the quasiperiodic tilings can be described by a quasilattice, which, contrary to a periodic lattice, consists of a quasi periodic sequence of two distinct intervals: unit interval ‘1’ combined with another, ‘τ’ interval with the width equal to (1+\sqrt{5})/2 for the pentagonal/decagonal case, and ‘1’ combined with ‘\sqrt{2}’ for the octagonal tiling. Typical quasiperiodic sequences are, e.g., 1, \sqrt{2}, 1, \sqrt{2}, 1, \sqrt{2}, 1, \sqrt{2}... and 1, τ, 1, τ, 1, τ, 1, τ... (Fig.28,34,35).

What are consequences of these discoveries for us? Whether it was Five Pillars of Islam and Solomon’s Seal or just a professional inquisitiveness, Islamic artists – as apparently the only ones – discovered and used principles of pentagonal/decagonal and octagonal quasiperiodicity in two-dimensional ornaments.

In the Eastern Islam, a tiling principle was dominant. The basic tiles – pentagons, butterflies, and rhombs with 72° vertices marked, have been a fairly used set of tiles, combined into 1D and 2D patterns with a subdued use of local 5-fold symmetry (Bourgoin 1973, Makovicky 1992, Lu and Steinhardt 2007). However, on the Blue Tomb of Maragha, NW Iran (1196-97 C.E.), they were combined into a half-cartwheel, 180° quasiperiodic pattern (the old artists appear to have always developed a cartwheel version of quasiperiodic patterns). A quasiperiodic pattern has a non-periodic profusion of local 5- and 10-fold configurations, which may or may not have a corresponding 5-fold internal symmetry (Fig. 27). However, if they are internally less (i.e., mirror-) symmetric, they can ‘rotate quasi-freely’ assuming all (two, five..) by pattern allowed positions. They can be, and were in Maragha, replaced by ornamental...
5-fold or externally 10-fold discs (rosettes). In the pattern from Darb-e-Imam (Esfahan, 1453 C.E.) the 10-fold discs are left unfilled as nodes of a large-scale net of pentagons (Lu and Steinhardt 2007). In these two occurrences, the cartwheel quasiperiodic discs have about the same extent and are embedded as ‘oversize’ nodes into a periodic pattern (cmm in the last case).

In the Western Islam, stress appears to be on quasilattices, decagonal and octagonal, respectively (Figs. 28-33). In the purest patterns (Makovicky et al. 1998, Makovicky and Fenoll 1996) they are outlined by white lines of tiles and they, in essential are star-studded quasilattices with background fields coloured according to the point group symmetry of the cartwheel pattern (Figs. 28 and 32). Whenever necessary for placing a star-like ornament, ‘phasons’, i.e., local sequence reversals of a type $1, \sqrt{2}, 1, 1, \sqrt{2}, 1, \sqrt{2}, 1, \sqrt{2}, 1...$ are freely performed.

For octagonal patterns, further development into much more complicated ornamental patterns is common: (rotated) centerpieces with different scale, rosettes, ornamental alterations (e.g., ‘averaging’ of adjacent 1 & $\sqrt{2}$ intervals, leaving out the white quasilattice lines, etc., with frequent departures from puristic octagonality (Fig. 33).

Dating –broadly about 1350 C.E. In Morocco, copies of decagonal patterns have been created practically until present times (Fig. 30), plus a long tiling-oriented development of octagonal patterns in which quasilattice is subdued in favor of 45°lozenge and square fields (‘tiles’). The frequently raised question whether the domes of muqarnas are an octagonal quasicrystalline tiling is treated in Figs. 38-41 with a negative answer.

**STRUCTURAL CLASSIFICATIONS**

Pure unit-mesh and plane-(layer-)group characterization gives only a partial picture of pattern variability. One of the further ways to divide the patterns, is to describe the type of Wyckoff positions occupied by individual elements, and the number of times a given (less special or quite general) Wyckoff position is occupied.

There exists another way of classifying patterns, a structural one, where we do not
have to define and select individual elements from the complex, often continuous pattern. Patterns can form series in which one pattern evolves into another, via a series of transitional patterns. Character of these changes determines the type of series (Makovicky 1989):

- **Expansion-reduction series** in which certain elements expand in area/importance at the expense of others which are being reduced;
- **Accretional series** in which a certain type of elements is multiplied without changing relationship to the rest of the pattern;
- **Intercalation series** in which additional elements are being introduced – intercalated – between the existing elements. They differ from the original ones. Omission series is the opposite sequence of changes.
- **Complication/simplification series** in which the existing elements receive or loose ornamentation appendages, change their shapes in a quantitatively expressible way – can be called (de)baroquization of the pattern;
- **Element-substitution series** in which a certain, increasing percentage of original, identical elements are being replaced by a new (e.g. differently coloured) element;
- **Pattern-reduction or crystallographic shear series** in which pattern is being reduced on selected crystallographically defined lines by a desired interval, connected with an exactly defined mutual shift of the pattern portions adjacent to such a line of reduction.

**Material and Style**

Material used has profound influence upon the patterns created and their symmetry. On the one end of spectrum are brickwork patterns in which one or several basic brick shapes and a limited number of sizes were used, eventually bevelled on edges when necessary. This characterizes the old, wonderful Seljuq brickwork of Central Asia, Iran and Turkey, with a plentiful use of brick ribs and recessed fields in order to achieve a play of sunlight on these uncoloured patterns (e.g., Makovicky 1989)(Figs. 43-47). Discovery of turquoise glaze led often to two-coloured, two-tier, interlacing patterns. At the other end of Islamic world, the marble-and-brick ornaments of Cordoba (about 936-1008 C.E.) represent another flowering of this style (Makovicky and Fenoll 1997)
Sadly, the technological advances in the field of glaze, e.g., as the *haft rengi* tiles of Safavid Iran, led to a substantial reduction in the level of two-dimensional ornamental crystallography, replaced by opulent floral designs.

The other end of the spectrum is represented by hand-fashioned colour-glazed ceramic tiles (zalij of Morocco), which allow virtually any shape and any symmetry – mostly ‘uncoloured’ or only dichroic, in spite of their often rich gamut of colours. Furthermore it is plaster (Fig. 19), less frequently stone, which also allow variations not normally observed in bricks. Here often limitation is dictated by the exposure of these elements to the weather (or, also, play of sunlight), a factor foreign to the smooth surface of ceramic tiles.

Different techniques are reflected especially in different statistics of plane group of symmetry used. Here we present diagrams for Cordoba (Makovicky and Fenoll 1997), a Seljuq Kharraqan Tower in Iran (Makovicky, in preparation) and the general statistics from Bourgoin (1973) who presents primarily ceramic tile patterns and some stucco patterns.

**KASBAH DE TEOUET**

In the mid-nineteenth century the family of El Glaoui’s were clan leaders controlling the age-old route from Marrakech to the Draa and Dades valleys; they were limited to their domain. However, in the hard winter of 1893, Sultan Moulay Hassan was passing on the way back from a disastrous military campaign and found himself at the mercy of Glaoui’s brothers for food and shelter for himself and his army. Their reception and care were handsomely rewarded: they became *caids* for the whole region and received lots of arms the sultan was forced to leave behind. By 1901 they were sole rulers of the area and when French occupied Morocco in 1912 they pledged their loyalty to the new rulers. They became *pashas of Marrakech* and *caids* for the region of Atlas and desert cities. As could be expected, they turned their income and power into building a residence appropriate to their new standing.
The kasbah of Telouet, built by them was abandoned only in 1956 but many of its dark-red earth buildings crumble fast as such constructions do without constant care. It is a labyrinth of rooms, doors and connecting passages. It is the reception rooms we come to visit, with iron-grilled windows, fine carved ceilings and zelij and plaster ornaments. Although they are 19th and early 20th century creations, they follow the traditional style and they are richer in character than the bulk of Marrakech ornaments of any age. They also give insight into lives of local warlords who, in spite of their new-found influence remained faithful to their safe, clan core area.

*Pay attention to the following* (and everything else you discover):

Any zero-dimensional ornaments and their symmetry?

Frieze ornaments and their true 1-D or truncated 2-D character..

Are the plaster ornaments one- or two-sided (plane- or layer groups of symmetry)?

Can you reconstruct the 2-D ornaments on the doors and similar objects?

Plane groups of zelij ornaments..

Relationship of site symmetry in plane- or frieze patterns and symmetry of the motif in this site..

Dichroic and polychromatic groups...?

Anatomy of octagonal ornaments: correlation/counterplay between a quasilattice and polygonal representation?

Anything interesting among the hangings?

Were they contrasting various symmetries for better effect?

Get ‘supervisors’ to take a look at your conclusions...

**Captions to the Figures**

**Fig. 1** Plane group of symmetry *cmm* with general positions and various types of special positions (on *m*, *2mm* and 2).

**Fig. 2** Pattern *p6* of raised bricks with general positions (d) and diverse special positions (a-c). Central Asia.

**Fig. 3** Dichroic plane groups *p4′g′m* (fig. b), *p4′gm′* (fig. c) and *p4g′m′* (fig. d) based
on the motif of the ‘maple leaf’ $p4gm$ (fig. a). Figs. a and b: Sala de la Barca, The Alhambra.

Fig. 4 Uncoloured and dichroic operations of symmetry (two-fold axes $2$, mirror planes $m$ and glide planes $g$.

Fig. 5 Selected dichroic plane groups derived from the plane group $pmm$.

Fig. 6 Selected dichroic plane groups derived from the plane group $cmm$.

(c and d): general and special positions, respectively, of the group $cm \ 'm$.

Fig. 7 Dichroic pattern in $p4'g'm$. Grand Mosque of Cordoba.

Fig. 8 Dichroic group $p4'g'm$. Special position on a neutral background. Shahrisabs, Uzbekistan.

Fig. 9 Dichroic group $pC4'4m$ derived from a $p4$ mosaic. Special positions on 4-fold axes and general positions.

Fig. 10 Dichroic group $pCm$. Baños de Alhambra.

Fig. 11 Four-coloured wave superposed on a dichroic pattern $p4'g'm$, which in turn is based on the $p4gm$ tiling. Patio de Mexuar, The Alhambra.

Fig. 12 Dichroic pattern $p4'g'm$ coloured by a three-coloured wave. Black tiles are nodes of the wave.

Fig. 13 Large-scale dichroic pattern $c'mm$ based on a $p4gm$ tiling.

Fig. 14 Pattern with a sequentially dichroic coloration. Stage 1: white/coloured; stage 2: coloured tiles $\rightarrow$ grey and black. Salon de Comares, The Alhambra.

Fig. 15 Interlaced version of the ‘maple leaf’ pattern. Layer group $p-4b2$.

Fig. 16 Pattern with a layer group $p422$.

Fig. 17 Interlaced Islamic pattern with a layer group $p622$.

Fig. 18 Pattern with 10-fold rosettes, layer group $c222$. A very common Islamic pattern.

Fig. 19 Square pattern $p4$, traced out in plaster. The Alhambra, Puerta de Vino (before 1300). Two types of stripes with a frieze group $pmm2$ situated between broader stripes $p1m1$ in two orientations.

Fig. 20 Another OD variant of the Puerta de Vino pattern, constructed by the action of two-fold axes situated in the $pmm2$ stripes.

Fig. 21 Islamic pattern based on the 14-fold rosettes. Stripes have frieze symmetry $p2$, with inter-stripe operations being either $m$ (resulting in a $pgm$ pattern) or 2 (resulting in a $cmm$ pattern). Alternatively, one can choose stripes with
symmetry close to $pmm2$ with two different inter-stripe operations (these being two different sets of two-fold axes).

Fig. 22 Intergrowth of two patterns based on 12-fold rosettes. Adopted from Bourgoin (1973).

Fig. 23 Statistics of plane groups of symmetry in Islamic patterns in the pattern collection of Bourgoin (1973).

Fig. 24 Statistics of plane groups of symmetry for the brick-and-marble patterns from the Great Mosque of Cordoba (Makovicky and Fenoll 1997).

Fig. 25 Statistics of one-dimensional groups of symmetry in the same mosque (Makovicky and Fenoll 1997).

Fig. 26 ‘Kite and dart’ version of a decagonal cartwheel quasicrystalline pattern. Five- and ten-fold discs and Conway worms are accentuated. Reinterpreted from Grünbaum and Shepherd (1987).

Fig. 27 A quasiperiodic tiling from the tomb tower of Maragha, NW Iran, with elements corresponding to those in fig. 26 accentuated by shading. Slightly simplified.

Fig. 28 A decagonal quasiperiodic cartwheel pattern. Note Ammann bars with widths equal to 1 and $\tau$, respectively, in a quasiperiodic succession. Museum of the Alhambra.

Fig. 29 Structure of the quasicrystalline pattern of the panel in the Museum of The Alhambra expressed as an assembly of PM1 decagons of Makovicky (1992), see Makovicky et al. (1998). Based on a free rotation of decagonal elements resulting in ‘averaged’ rosettes.

Fig. 30 Quasicrystalline decagonal cartwheel pattern from Palais Royal, Fez, Morocco.

Fig. 31 A $cmm$ pattern of decagonal rosettes. The Alhambra and elsewhere.

Fig. 32 Quasilattice of the octagonal quasiperiodic pattern of the Patio de las Doncellas, Los Reales Alcazares, Sevilla. Width of intervals is unity and $\sqrt{2}$, respectively.

Fig. 33 An octagonal pattern from the Salon de Comares, The Alhambra, with a separate central disc and frequent phason shifts (frequently interchanged unit and $\sqrt{2}$ stripes).

Fig. 34 Primary Ammann bars superimposed upon an aperiodic octagonal tiling. Modified from Socolar (1989).
Fig. 35  Secondary Ammann bars superimposed on the same tiling as in Fig. 34.

Fig. 36  Quasicrystalline octagonal tiling with Ammann’s vertex and edge decorations and with a system marking lines derived from them by Makovicky and Fenoll (1996).

Fig. 37  A quasiperiodic pattern with a system of marking lines superimposed; its configuration indicates quasiperiodicity.

Fig. 38  The dome of the Sala de dos Hermanas in the Alhambra. Bold lines indicate two systems of ornamental decorations incorporated in the assembly of adarajas (wooden tiles of the stalactite vault). Makovicky and Fenoll (1996).

Fig. 39  Transcription of tiles of the dome of La Sala de Dos Hermanas into the square-and-lozenge octagonal tiling. Grey elements: ‘rotating octagons’ which allow a multiplicity of orientational interpretations.

Fig. 40  Marking lines applied to the (transcribed) dome of La Sala de Dos Hermanas. Their configuration indicates frequent departures from quasiperiodicity.

Fig. 41  A system of Ammann bars (unity [white] and √2 [grey]) applied to the tiling of the dome of La Sala de Dos Hermanas. Note occurrences of √2-√2 and multiple 1-1-1 contacts not allowed in a quasiperiodic octagonal tiling.

Fig. 42  Three inflation stages of an octagonal quasiperiodic tiling of squares-and-lozenges.

Fig. 43  Pattern of raised bricks, Djami Mosque in Gurgan, N Iran (12th century). Plane group p2 resulting from affine transformation of a p4 pattern.

Fig. 44  A large-scale p4mm pattern of raised bricks from the caravanserail Ribat-e-Sharaf, Iran.

Fig. 45  A p4gm pattern of raised bricks. Minaret of the Qalan Mosque, Bukhara, Uzbekistan (Seldjuk, 1127).

Fig. 46  A complex swastika pattern of raised, stacked bricks, plane group p4. Gunbad i’Alavyian, Hamadan, 13th century.

Fig. 47  A p6mm pattern of raised, partly fashioned bricks. Tympanum of a tomb-tower at Kharraqan, Iran, 1067-68.

Fig. 48  (a through c) A homologous series of patterns based on swastikas and squares with single, double and triple separation lines, respectively. The Grand Mosque of Cordoba.

Fig. 49  Crystallographic shear applied to a square pattern. Upper parts show the
procedure, the resulting configuration is shown in the lower portions of the figure. Grand Mosque of Cordoba.

Selected References


Fig. 23

Total number of classified patterns: 200

Fig. 24

Total number of classified patterns: 42

Fig. 25

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Total number of studied cases: 46
Kaleidoscope of Moroccan ornamental art - a mathematician's view

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Kaleidoscope of Moroccan ornamental art - a mathematician's view

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First part: overview of the geometric decorative style in traditional Moroccan architecture, and some of its connections to contemporary mathematics.

1. Generals concepts and examples. 2D or 3D decoration: Zellij panels and muqarnas structures. Different families in terms of main symmetry (octagonal, triangular or pentagonal...).
2. Bidimensional decoration: the art of Zellij. The octagonal family, a morphogenetic point of view. Elements of design technique. Relations with the mathematics notions of self-similarity and octagonal quasicrystals.
3. Tridimensional decoration: the muqarnas structures, a modular technique. The main units. The "àmap": a system of plane representation. The octagonal family. Examples with alternative symmetry. Relations with octagonal quasicrystals.

Second part: workshop.

1. Immersion in the art of Zellij with the "Zellij multipuzzle". 3 groups of 4 persons working together.
2. From printed examples, learn to recognize different variations of the same pattern, its general structure (the "skeleton"), and its possible irregularities.
3. Working with muqarnas. Experimentations with units. Drawing a map. Interpretation of a part of octagonal plane quasicrystal in terms of muqarnas.
Artist's and artisan's approach to, and practical realization of, Islamic ornaments

Mohamed Jama Eddine BENATIA

Artist / Craftsman
Artist's and artisan's approach to, and practical realization of
Islamic ornaments

Mohamed Jamal Eddine Benatia, Abdelaziz JALI, Abdelmalek THALAL

Introduction

Two types of Moroccan ornamental art are found on diversity of materials, tiles, bricks, wood, brass, plaster and may types of objects, plane ornamental art and tri-dimensional art. The first type is geometric patterns called “Tastir” (fig 1) occur in rich profusion throughout Moroccan art.

Figure 1

Another distinct plane pattern type perfected in Moroccan art is the arabesque. This comprises curvilinear elements resembling leafed and floral forms called “Taouriq” fig2 and Tachjir fig3. In such pattern spiral forms intertwine, undulate and coalesce in continuously.

Taouriq

Tachjir

The tri-dimensional ornamental art is called Moukarnas
In this presentation we are only interesting in the plane ornamental art, particularly in the geometric drawing or “Tastir”. We describe the methods of construction of the geometric patterns encountered in the Islamic art. We focus on the Moroccan method which has the particularity to respect scrupulously the rules which were adopted by the “Maâlam” (master craftsmen) and handed over to their disciples.

Two methods of construction used in the realization of Moroccan geometric patterns are presented. These methods based on the concept of symmetry can be adapted to any material shape (plaster, wood, metal, marble, … ). They consist in tracing a grid with precise criteria of measurement called Hasba. The framework used to draw the grid is generally square and rectangular. Patterns are generated by the traced grid.

1 - The first method or random method

This method called also random method used by craftsmen in the Islamic Eastern is not governed by any rules. It consists in tracing circle inside any square framework. The figure is then divided by lines traced in certain directions in order to construct a grid which generates the final pattern. There is no external border induced by construction of the pattern fig4.

Figure 4
On the other hand the Maâlem in the Islamic Western adopted two mainly methods based on accurate rules, the first one adapted to construct the finer mosaics or “zellijes” panel is called “Foussâïfissa” method, the second method uses the notion of module measure or “Hasba”.

2 - **Fousaïfissa method**

The method consists in drawing a framework generally square. The grid is resulting from by the intersection of vertical, horizontal and diagonal parallel lines. The Zqaq or alley is defined by 2 spaces as shown on the fig5. The length of the framework is perfectly determined and so it is the border. The type and the symmetry of the resulting pattern depends on the division of the square sides.

This method adapted to the construction of the finer mosaics adopts a unit of measure called Zqaq (alley). It generally leads to a multiple of 8-fold symmetry. The decorated framework space or basic pattern is constituted of central area called Naâoura (rosette or star), a peripheric area at the limit of the framework or Àach, and an interface area Hizam (belt) between them. Rosette, with 8-fold to 96 - fold symmetry or more, is the most eye-attracting element. It often hides the imperfections in the other areas. However the accurate construction of belt and periphery remains essential for the harmony and the artistic value of the pattern.

This method introduces a misfit between central area and periphery. For rosette with symmetry great than 16-fold craftsmen have to transcend the rule to obtain the compatibility between the rosette and the border of the framework. So they break the symmetry of the pattern in the interface area.

The perfect adequacy between the three areas Figure 6 requires precise rules of construction. To achieve the panel in other materials like wood or plaster, the Moroccan craftsmen use the method based on the measure module or Hasba.

![Figure 5](image_url)
3 - “Hasba method”

To build valid motifs, designers have to respect scrupulously several artistic rules of drawing based on the notion of module measure. They define the framework which is generally square, rectangle Figure 7, octagon and triangle are not uncommon and trace a grid with precise criteria of measurement called Hasba.
3-1 Module Measure or Hasba

The strip which borders the framework is called “Lahsor”. Its width zqaq(alley) represents the measure module of the framework (Figure 8).

The ratio of the length of the framework to “zqaq” is essential for the achievement of good patterns which respect artistic criteria. Such ratio which is used empirically by craftsmen masters to achieve their patterns can be an integer or a fraction number.

The median line of the border is a mirror which repeats the basic motif in the two directions of the plane.

3-2 Interlacing ribbons “L’qataa wal’maqtaa”

In finer design the line are transformed into ribbon “l’qtib” endlessly interlaced. Each qtib alternatively crosses or is crossed by another qtib (Figure 9).
L’aqtib is an **infinitely continuous** strip having a constant width. It is defined as the quarter of zqaq. Its width is determinant for the achievement of patterns. In addition of their aesthetic function the interlacing ribbons are an efficient check on the artistic validity of patterns. If l’aqtib is stopped somewhere then the pattern is wrong or markouse (Figure 10).

3-3 The corners of patterns - Khatem Slimani (Seal of Solomon)

We find in almost all patterns a in the four corners of the pattern. Some patterns can be extended and generate one or several patterns which obey to the construction criteria of patterns. The large pattern obtained is called application (Figure 11).
3- 4 Name of pattern

The name of pattern is related to the symmetry of the rosettes or stars locate at the center of pattern. Rosette, with n-fold symmetry, is the most eye-attracting element. It is composed of n divisions called “safts”.
For instance in the figure 11 the is called “Sattachri” or 16-fold.

Conclusion

Relatively recently, several authors have published large collections of Islamic patterns. They have presented their own analysis of the method of constructions. The insights offered by all of them are interesting and valuable but they don’t explain how did the Islamic patterns evolve from the simple to the complex? Furthermore the regularity and the perfect symmetry of the patterns hide, without doubt, mathematical algorithms developed by the ancient designers and used now in empirical away by the modern maâlems.
The future prospects aim, with the help of mathematicians, to understand the algorithm used by designers and to express the empirical method of “hasba” into mathematical model. It will be then possible to achieve all the known patterns on any plane area whatever by using computational program, to improve the existing pattern and to innovate others 2-dim as well as 3-dim patterns.