Color Symmetry

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$a$ : 90°-counterclockwise rotation about $O$

$b$ : mirror reflection in the horizontal line through $O$
\( G = \{ e, a, a^2, a^3, b, ab, a^2 b, a^3 b \} \)

\( G = \langle a, b \rangle \cong D_4 \) (or \( 4m \)), a dihedral group of order 8
If $x \in \mathbb{R}^2$, $g \in G$, $gx$ refers to the image of $x$ under the isometry $g$.

$Gx := \{gx \mid g \in G\}$ is called the $G$-orbit of $x$. 

Diagram: A point $x$ is marked on the intersection of the lines $a^2 b$ and $b$. Other lines include $a b$, $a^3 b$, and $a^2 b$. The origin $O$ is at the center.
If $x \in \mathbb{R}^2$, $g \in G$, $gx$ refers to the image of $x$ under the isometry $g$.

$Gx := \{gx \mid g \in G\}$ is called the $G$-orbit of $x$. 
The $G$-orbit of a point $x \in \mathbb{R}^2$ if $x$ does not lie on any of the mirror axes.
The $G$-orbit of $x$ if $x$ is on one of the mirror lines and $x \neq O$
The symmetry group of the configuration of points on the left is

\[ G = \{ e, a, a^2, a^3, b, ab, a^2 b, a^3 b \} . \]
Every \( g \in G \) results in a permutation of the colors in the set \{B, G, R, Y\}. The coloring is called *perfect*. 

\[
a : (R \ Y \ G \ B)
\]

\[
b : (Y \ B)
\]

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\]

\[
b : (Y \ B)
\]
Let $S < G$. Every partition $P$ of $X = Gx$ of the form $P = \{gSx \mid g \in G\}$ gives a perfect coloring of the eight-point configuration.

Two points are assigned the same color if and only if they belong to the same set in the partition $P$.

The number of colors in the coloring is given by $[G : S]$, the index in $G$ of the subgroup $S$.

$[G : S]$ is the number of distinct left cosets $gS$, $g \in G$. 

The coloring on the left uses the subgroup $S = \{e, b\}$.

The four distinct left cosets of $S$ in $G$ are

1. $S = \{e, b\}$
2. $aS = \{a, ab\}$
3. $a^2S = \{a^2, a^2b\}$
4. $a^3S = \{a^3, a^3b\}$

The partition $P = \{Sx, aSx, a^2Sx, a^3Sx\}$ corresponds to the given coloring.
<table>
<thead>
<tr>
<th></th>
<th>ax</th>
<th>$a^2x$</th>
<th>$a^3x$</th>
<th>bx</th>
<th>abx</th>
<th>$a^2bx$</th>
<th>$a^3bx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|c|}
\hline
S_x & aS_x & a^2S_x & a^3S_x \\
\hline
\{1, 5\} & \{2, 6\} & \{3, 7\} & \{4, 8\} \\
\text{red (R)} & \text{yellow (Y)} & \text{green (G)} & \text{blue (B)} \\
\hline
\end{array}
\]

\[P = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}\]

\[
\begin{align*}
2 & \quad 6 \\
\text{\textbullet} \quad \text{\textbullet} \\
7 & \quad \text{x} \quad 1 \\
3 & \quad \text{\textbullet} \quad 5 \\
8 & \quad 4 \\
\end{align*}
\]
Lattice diagram of subgroups of
\[ G = \{ e, a, a^2, a^3, b, ab, a^2b, a^3b \} \cong D_4: \]

\[
\begin{array}{c}
G \\
\{ e, a^2, b, a^2b \} \{ e, a, a^2, a^3 \} \{ e, a^2, ab, a^3b \} \\
\{ e, b \} \{ e, a^2b \} \{ e, a^2 \} \{ e, a^3b \} \{ e, ab \} \\
\{ e \} \\
\end{array}
\]
Identify which subgroup $S$ of $G$ was used for each of the following colorings:
Consider the partition of $X = Gx$ given by the set

$$\{\{1, 5\}, \{2, 8\}, \{3, 7\}, \{4, 6\}\}.$$
Not every element of $G$ permutes the 4 colors (the $90^\circ$-rotation $a$, for example).

The coloring is not perfect.

The elements of $G$ which effect color permutations are $e$, $a^2$, $b$, and $a^2 b$. 
<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>$e$</td>
<td>$I$ (identity permutation)</td>
</tr>
<tr>
<td>$a^2$</td>
<td>$(R \ G)(B \ Y)$</td>
</tr>
<tr>
<td>$b$</td>
<td>$I$</td>
</tr>
<tr>
<td>$a^2b$</td>
<td>$(R \ G)(B \ Y)$</td>
</tr>
</tbody>
</table>
The symmetries in $G$ which effect a permutation of the colors of the colored pattern are called color symmetries and they form a subgroup $H$ of $G$ called the color symmetry group or color group.

The elements of $H$ resulting in the identity permutation (of colors) form a subgroup $K$ of $H$ called the color fixing group.
Let $C$ be the set of colors of the colored pattern and $S_C$ be the group of permutations of $C$.

If $H$ is the color group, a homomorphism

$$f : H \rightarrow S_C$$

is induced where if $h \in H$, $f(h)$ is the permutation of $C$ effected by $h$.

The color fixing group $K$ is the kernel of the homomorphism $f$.

By the Fundamental Isomorphism Theorem for groups, $H/K \cong f(H)$. $H/K$ is called the color permutation group.
color group:
\[ H = \{ e, a^2, b, a^2 b \} \]

color fixing group:
\[ K = \{ e, b \} \]

color permutation group:
\[ H/K \cong \langle (Y \ B) \rangle \cong \mathbb{Z}_2 \]
If \( y \neq O \) is a point on the mirror line of reflection \( b \), the figure on the right is the \( G \)-orbit of \( y \).

\[ Gy = \{ y, ay, a^2y, a^3y \} \]

Note: \( by = y \), \( aby = ay \),
\[ a^2by = a^2y, \, a^3by = a^3y \]
The *stabilizer of y in G*, denoted by \( \text{Stab}_G(y) \), is the set \( \{ g \in G \mid gy = y \} \). It is also called the *site symmetry group of y*.

\[
\text{Stab}_G(y) = \{ e, b \}
\]

By the Orbit-Stabilizer Theorem:

\[
|Gy| = [G : \text{Stab}_G(y)] = \frac{|G|}{|\text{Stab}_G(y)|} = \frac{8}{2} = 4
\]
Every partition of $Gy$ of the form 

$$P = \{gSy \mid g \in G\},$$

where $\text{Stab}_G(y) \leq S \leq G$, gives rise to a perfect coloring of the configuration of four points.

Since $\text{Stab}_G(y) = \{e, b\}$, the choices for $S$ are $\{e, b\}$, $\{e, a^2, b, a^2b\}$ and $G$.

The number of colors in the corresponding coloring is respectively 4, 2, 1 and is given by the index $[G : S] = \frac{|G|}{|S|}$. 

The configuration of four points is illustrated below:

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & y \\
\bullet & \bullet & \bullet \\
\end{array}
\]
Perfect colorings of the 4-point configuration using 4, 2, and 1 color
$H = \{ e, a^2, ab, a^3 b \}, \ K = \{ e, ab \}$

This coloring is determined by the partition \( \{ Ky, a^2 Ky \} \) where \( K = \{ e, ab \} \).
In all the examples shown, the following result was used:

**Theorem**

Let $X$ be a set, $x \in X$, and $H$ a group that acts transitively on $X$ ($X = Gx$). If $P$ is a partition of $X$ on which $H$ acts transitively, then

$$P = \{ hSx \mid h \in H \}$$

where $\text{Stab}_H(x) \leq S \leq H$. 
$c : 60^\circ$-counterclockwise rotation about the center

$d :$ mirror reflection about the horizontal line through center
\[ G = \langle c, d \rangle = \{ e, c, c^2, c^3, c^4, c^5, d, cd, c^2d, c^3d, c^4d, c^5d \} \cong D_6 \]
$G$-orbit of point 1:
$Y = \{1, 2, 3, 4, 5, 6\}$

$G$-orbit of point 7:
$Z = \{7, 8, 9, 10, 11, 12\}$

$X = Y \cup Z = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, the set of points in the configuration
The coloring corresponds to the partition

\[ P = \{\{1, 4, 9, 12\}, \{2, 5, 7, 10\}, \{3, 6, 8, 11\}\}. \]
Every \( g \in G \) permutes the 3 colors.

The coloring is perfect.
Stabilizer in $G$ of red
$= J = \{ e, c^3, d, c^3d \}$

$P = \{ gJ \{1, 9\} \mid g \in G \}$
where $J \{1, 9\} = J1 \cup J9$

$[G : J] = 3$, the number of colors
Let $G_0 = \{ e, c, c^2, c^3, c^4, c^5 \}$, the set of rotations in $G$.

The same coloring is also given by $P = \{ gJ_0 \{1, 9\} \mid g \in G_0 \}$ where $J_0 = \{ e, c^3 \}$. 
A non-perfect coloring of $X$ where

$$H = K = \{e, c^2, c^4, d, c^2d, c^4d\} \cong D_3$$
The coloring corresponds to the partition

\[ P = \{ H1, H\{2,7\}\} = \{\{1, 3, 5\}, \{2, 4, 6, 7, 8, 9, 10, 11, 12\}\} \]

where \( H\{2,7\} = H2 \cup H7 \).
A 5-coloring of the vertices of a regular dodecahedron
\[ a : (1 \ 2 \ 3 \ 4 \ 5) \]
\[ b : (2 \ 5 \ 4) \]
\[ ab : (1 \ 2)(3 \ 4)(5) \]
color group: $H = \langle a, b \rangle$
$\cong 532 \cong A_5$, $|H| = 60$
If $S = \text{Stab}_H 2$, then $S \cong A_4$, $|S| = 12$, and $[H : S] = 5$
partition $P = \{ hSv \mid h \in H \} = \{ Sv, aSv, a^2Sv, a^3Sv, a^4Sv \}$
A 5-coloring of the faces of a regular icosahedron
A 6-coloring of the edges of a regular dodecahedron

\[ \begin{align*}
P &= \{A, B, C, D, E, F\} \quad \text{where} \\
A &= \{1, 8, 25, 27, 12\} \\
B &= \{2, 9, 21, 28, 14\} \\
C &= \{3, 10, 22, 29, 16\} \\
D &= \{4, 6, 23, 30, 18\} \\
E &= \{5, 7, 24, 26, 20\} \\
F &= \{11, 13, 15, 17, 19\}
\end{align*} \]
A 6-coloring of the edges of a regular dodecahedron

\[ P = \{ A, B, C, D, E, F \} \] where

\[
\begin{align*}
A &= \{ 1, 8, 25, 27, 12 \} : \text{orange} \\
B &= \{ 2, 9, 21, 28, 14 \} : \text{violet} \\
C &= \{ 3, 10, 22, 29, 16 \} : \text{blue} \\
D &= \{ 4, 6, 23, 30, 18 \} : \text{green} \\
E &= \{ 5, 7, 24, 26, 20 \} : \text{pink} \\
F &= \{ 11, 13, 15, 17, 19 \} : \text{yellow}
\end{align*}
\]
A 6-coloring of the edges of a regular dodecahedron

\[
\begin{align*}
  a & : (A \ E \ D \ C \ B)(F) \\
  b & : (A \ F \ B)(C \ D \ E) \\
  ab & : (A \ F)(B \ E)(C)(D) \\
  c & : (A \ D)(B \ C)(E)(F)
\end{align*}
\]
A 6-coloring of the edges of a regular dodecahedron

Color group: $H = \langle a, b \rangle$

$\cong 532 \cong A_5$, $|H| = 60$

Let $S = \langle a, c \rangle$. Then

$S \cong 522 \cong D_5$, $|S| = 10$, and

$[H : S] = 6$

The coloring is given by the partition

$P = \{ hS15 \mid h \in H \}$.
Color Symmetry
Color Symmetry
a : $60^\circ$-counterclockwise rotation about $\theta$

$r :$ mirror reflection about horizontal line through $\theta$
\[ G \cong \langle u, v, a, r \rangle \cong p6m \]
\[ S \cong \langle u^3, u^2v, a, r \rangle \cong p6m, \quad [G : S] = 3 \]
\[ P_1 = \{S_x, S_y, S_z\} \]
\[ P_2 = \{S_x, uS_x, u^2S_x\} \]
\[ P_3 = \{S_y, uS_y, u^2S_y\} \]
\[ P_4 = \{S_z, uS_z, u^2S_z\} \]
$P_1 = \{S_x, S_y, S_z\}$
\[ P_2 = \{ Sx, uSx, u^2Sx \} \]
\[ P_3 = \{ S_y, uS_y, u^2 S_y \} \]
$P_4 = \{ Sz, uSz, u^2Sz \}$
A 4-coloring of the points of a hexagonal lattice
$a: 60^\circ$-counterclockwise rotation about $\theta$

$r: \text{mirror reflection about the horizontal line through } \theta$
$G = \langle u, v, a, r \rangle \cong p6m$

Let $L = \langle u^2, v^2 \rangle$

Partition $P = \{ L\theta, uL\theta, vL\theta, uvL\theta \}$
\[ G = \langle u, v, a, r \rangle \cong p6m \]

color group: \( H = G \)

color fixing group:
\[ K = \langle u^2, v^2, a^3 \rangle \cong p2 \]

\[ [G : K] = 4 \cdot 6 = 24 \]

color permutation group: \( G/K \cong S_4 \)

glide reflection \( a^2ru : (B \ Y \ R \ G) \)
A perfect precise 6-coloring of the triangles in the regular tessellation \((3^6)\)
We can obtain a torus by pairing the edges of the yellow hexagon.
A $p4m$ (or $*442$) pattern
Subpatterns obtained from the $p4m$ pattern using various index-2 subgroups

- pmm (*2222)
- p4 (442)
- cmm (2*22)
Subpatterns obtained from the \( p4m \) pattern using various index-2 subgroups

p4m \((*442)\)  p4g \((4*2)\)
Subpatterns obtained from the $p4m$ pattern using various index-2 subgroups

\[ p4g \ (4*2) \quad p4m \ (\ast442) \]
A triangle group $H$ of type $*pqr$ ($p, q, r \geq 2$) has seven index 2 subgroups if all of $p, q,$ and $r$ are even namely $pqr, *p(r/2)p(q/2), *q(p/2)q(r/2), *r(p/2)r(q/2), p \ast (r/2)(q/2), q \ast (p/2)(r/2)$ and $r \ast (p/2)(q/2)$. The group $H$ has three index 2 subgroups if exactly one of $p, q, r$ is odd and one index 2 subgroup if at least two values of $p, q, r$ are odd.

By the above theorem, the group $*442$ (or $p4m$) has 7 index-2 subgroups, and hence the 7 subpatterns shown.
flat origami tessellations
a mountain fold
(indicated by a solid line)

a valley fold
(indicated by a broken line)
An example of a *crease pattern*
René P. Felix  
Color Symmetry

1

2

front

back
René P. Felix  

Color Symmetry

1

2

front  back

What does the Japanese art of paper folding have to do with higher math? Plenty. Erik Demaine’s origami work provides insights as readily into the problems of sheet-metal engineering as it does into those of robotics and molecular biology.

Now he’s tackling the hottest folding problem of the day: finding the rules that govern how protein molecules twist into the complex shapes that are key to their biological function. Predicting how they do that would help pharmaceutical firms design more effective drugs.
end