Translational tilings of a convex body, with multiplicity

Workshop on Mathematical Crystallography, Manila

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Joint work with Nick Gravin and Dmitry Shiryaev
NTU, Singapore

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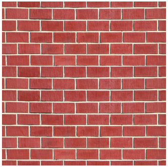
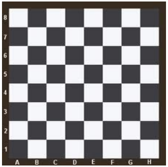
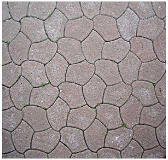
Tilings in our daily life

Tilings are everywhere around us: Bee hives, chessboards, brick walls, and so on.







They can be somewhat more complicated...

We can use several different tiles and get completely aperiodic tiling in a simple way:

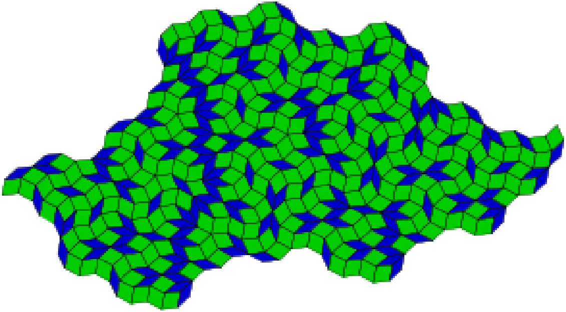


Figure: Penrose tiling

They can be somewhat more complicated...

If there is only one tile, but we allow to rotate it, we can get this:

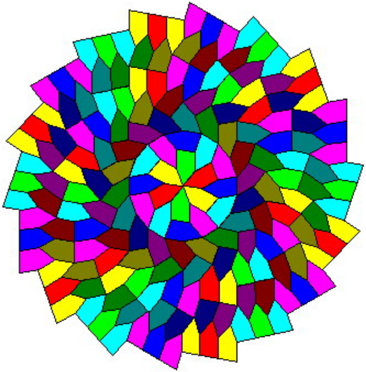


Figure: The Hirschhorn tiling

General to specific

We consider first a very specific case, namely translational tilings by a convex object:

- We have only **one** tile
- Our tile is a **convex** object (in fact, it is necessarily a polytope)
- We tile by a set of (discrete) **translation** vectors
- We tile the **whole space** $\mathbb{R}^d$, so that every point gets covered exactly once.

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Indicator functions of sets

Definition

Given any set $P \subset \mathbb{R}^d$, we let

$$1_P(x) := \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{if } x \notin P. \end{cases}$$

What is a tiling, formally?

Definition

We say that P tiles $\mathbb{R}^d$ with the discrete set of vectors Λ, if

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = 1,$$

for all $v \notin \partial P + \Lambda$.

What is the question?

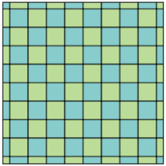
- What can we say about a polytope P that tiles the whole space by translations?
- Can we classify all such polytopes somehow?

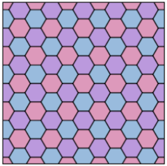
First steps: $\mathbb{R}^2$

Ideas:

- The tile is a convex polygon
- It should by centrally symmetric
- It should have some ”nice” angles

Parallelograms and regular hexagons are the only convex bodies that tile $\mathbb{R}^2$ by translations.





First steps: $\mathbb{R}^3$

The first non-trivial case is dimension 3. The first results we obtained by Russian crystallographer Fedorov.

Fedorov classified all of the combinatorially distinct polytopes that tile $\mathbb{R}^3$.

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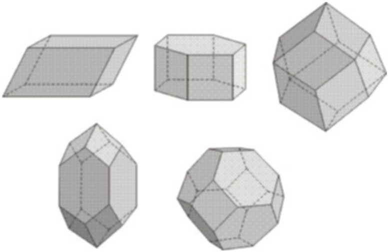
First steps: $\mathbb{R}^3$

Theorem (Fedorov, 1885)

There are five different combinatorial types of convex bodies that tile $\mathbb{R}^3$:



Big question:

What happens in higher dimensions? Can we “classify” all polytopes that tile $\mathbb{R}^d$ by translation? (Answer is somehow “yes” and “no”, depending on the definition of ”classify”....)

Minkowski and his results

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Theorem (Minkowski, 1897)

If a convex polytope P tiles $\mathbb{R}^d$ by translations, then

- *P is centrally symmetric*
- *Every facet of P is centrally symmetric*

Corollary

*Every convex polytope that tiles $\mathbb{R}^1$, $\mathbb{R}^2$ or $\mathbb{R}^3$ is a **zonotope**!*

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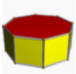
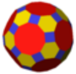
Zonotope? What's that?

Definition

A **zonotope** in $\mathbb{R}^d$ is, equivalently:

- polytope, all of which faces are centrally symmetric
- A Minkowski sum of several line-segments
- A projection of higher-dimension cube





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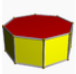

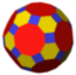

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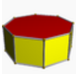

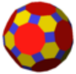

The 24-cell: a 4-dimensional counterexample

There exist 4-dimensional polytopes which tile $\mathbb{R}^4$, but are not zonotopes.

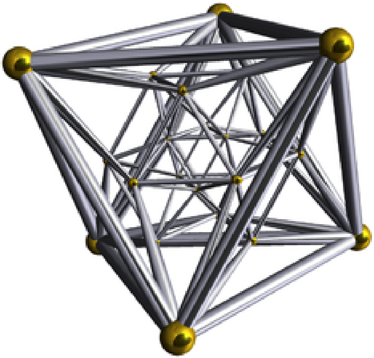

Figure: 24-cell (really, its projection into $\mathbb{R}^3$)

Venkov-McMullen

It took 50 years to find the missing link (a converse) for the characterization.

Theorem (Minkowski, 1897; Venkov, 1954; McMullen, 1980)

A convex polytope tiles $\mathbb{R}^d$ by translations if and only if

- *it is centrally symmetric*
- *each of its facets are centrally symmetric*
- *each belt contains 4 or 6 co-dimension 2 faces*

It gives us 52 tilers of $\mathbb{R}^4$, several thousand for $\mathbb{R}^5$...

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Tilings with multiplicities

A natural generalization of a tiling is a **tiling with multiplicity** k (k-tiling).

Definition

We say that P k-tiles $\mathbb{R}^d$ with a discrete set of translation vectors Λ, if

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

for all $v \notin \partial P + \Lambda$.

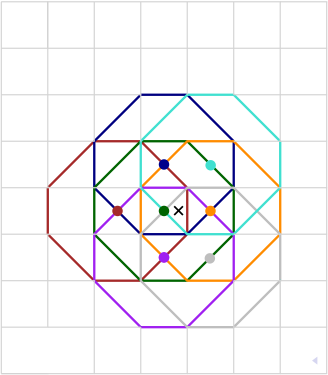
If $k = 1$ this becomes a usual tiling.

Example: The regular octagon

For example, take a regular octagon.

As we already know, it **does not tile** $\mathbb{R}^2$.

However, it **does** tile $\mathbb{R}^2$ with multiplicity 7, and with $\Lambda = \mathbb{Z}^2$!



Some known results on k-tilings

Bolle (1994) gave a characterization for lattice k-tilings of $\mathbb{R}^2$ in terms of distances between vertices of a polygon.

Kolountzakis (2000) proved that for every k-tiling of $\mathbb{R}^2$, Λ is union of finite number of lattices.

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Our main result

Theorem (Gravin, Robins, Shiryaev, 2011)

If a convex polytope P k-tiles $\mathbb{R}^d$, then

- *P is centrally symmetric*
- *Every facet of P is centrally symmetric*

A partial converse

Theorem (Gravin, Robins, Shiryaev, 2011)

*Moreover, if P is **rational**, then these two conditions are also sufficient.*

Counting Λ-points inside P

If P k-tiles $\mathbb{R}^d$ with the set of translations Λ, then for every general position of $-P$ in space, there are exactly k points of Λ in the interior of $-P$.

The reason for this is simple:

$$\sum_{\lambda \in \Lambda} 1_{-P+v}(\lambda) = \sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

since $\lambda \in -P + v$ if and only if $v \in P + \lambda$.

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Solid angles come in too, to play an equivalent role

This observation can be generalized into an interesting equivalence:

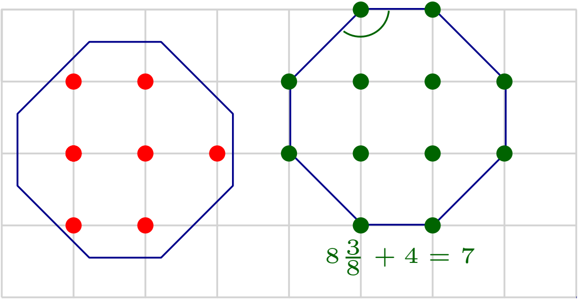
Theorem (Gravin, Robins, Shiryaev)

A polytope P k-tiles $\mathbb{R}^d$ with the discrete set of translations Λ if and only if

$$\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda) = k,$$

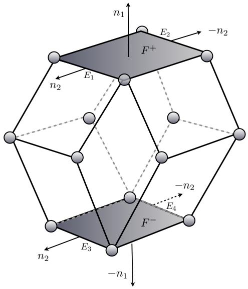
for every $v \in \mathbb{R}^d$.

Example: regular octagon that 7-tiles $\mathbb{R}^2$



Step 1. Counting Λ-points inside opposite facets

We outline the methods of proof, only for the first part of the main result. Suppose that F^+ and F^- are facets of P with normals n_1 and $-n_1$. P is in general position w.r.t. n_1 if there are no Λ-points on the boundaries of F^+ and F^-. We then conclude that the number of Λ-points inside F^+ and F^- are equal.



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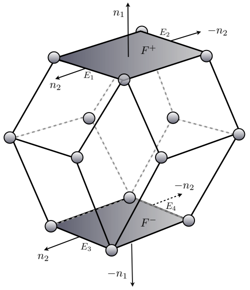

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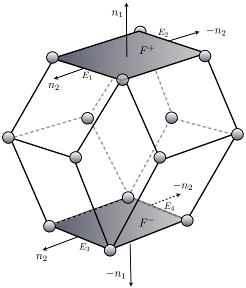

Step 2. From counting Λ-points to counting volumes

Next, we can show that the volumes of F^+ and F^- are equal, using asymptotic arguments.
(this is the technically difficult part)

Step 3. Minkowski strikes again

Now we recall a wonderful theorem of Minkowski's:

Theorem

A convex polytope in $\mathbb{R}^d$ is uniquely defined by its facet normals and its facet $(d-1)$-volumes.

But P and $-P$ have the same facet $(d-1)$-volumes, so $P = -P$, and therefore we reach the desired conclusion that P is centrally symmetric.

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The Harmonic Analysis approach

We recall that for any integer k, the polytope P k-tiles $\mathbb{R}^d$ with a lattice L, if:

$$\sum_{\lambda \in L} 1_P(\lambda - v) = k,$$

for all $v \notin \partial P + L$. We notice now that the left-hand side is a periodic function of v, where the period is a fundamental parallelepiped of L. It therefore has a Fourier expansion on this domain. The Poisson summation formula now gives us this Fourier expansion, as follows.

Poisson summation

Given any “nice” function f on $\mathbb{R}^d$, we have

$$\sum_{n \in L} f(n) = \sum_{\xi \in L^*} \hat{f}(\xi),$$

where by definition $\hat{f}(\xi) := \int_{R^d} f(x) e^{2\pi i \langle x, \xi \rangle} dx$, and where the dual lattice is defined by $L^* := \{x \in \mathbb{R}^d \mid \langle l, x \rangle \in \mathbb{Z}, \text{ for all } l \in L\}$.

Poisson summation

One way to think about the dual lattice is by recognizing it as the kernel of a very natural character acting on the lattice L. For each $l \in L$, let

$$\Phi_L(x) := e^{2\pi i \langle l, x \rangle}.$$

The kernel of this character Φ is
$L^* := \{x \in \mathbb{R}^d \mid \langle l, x \rangle \in \mathbb{Z}, \text{ for all } l \in L\}$.

The Harmonic Analysis approach

Thus, by Poisson summation, we have

$$k = \sum_{\lambda \in L} 1_P(\lambda - v) = \frac{1}{|\det L|} \sum_{m \in L^*} \hat{1}_P(m) e^{-2\pi i \langle v, m \rangle}.$$

Now we use the fact that Fourier series expansions are unique, so all the nonzero Fourier coefficients on the right must vanish, because k is constant. We have thus been naturally lead to the proof of another fascinating equivalence for any k-tiling.

The Harmonic Analysis approach

Lemma

A convex polytope P k-tiles $\mathbb{R}^d$ by translations with the lattice L if and only if both of the following conditions are true:

- $\hat{1}_P(m) = 0$, *for all nonzero vectors* $m \in L^*$, *the dual lattice of* L.
- $k = \frac{Vol(P)}{|\det(L)|}$.

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The Harmonic Analysis approach

We expand on the second part of this Lemma. When we compute $\hat{1}_P(0)$, we get:

$$\hat{1}_P(0) = \int_{R^d} 1_P(x) e^{(2\pi i)(0)} dx = \int_{1_P} dx = Vol(P).$$

Therefore, comparing the constant terms of both sides of the Poisson summation formula, we get $k = \frac{Vol(P)}{|\det(L)|}$.

The Harmonic Analysis approach

Thus, we must study the vanishing of the Fourier transforms $\hat{1}_P(m)$, also called the frequency spectrum. [relevant research: Mihalis Kolountzakis, Tim Gowers, Terry Tao, Mate Matolci, Alex Iosevich, Izabella Laba, ...]

What's next? Open questions

- The most interesting question: Give an analogue of the Venkov-McMullen converse for k-tilings.
- Given k, describe all polytopes that k-tile.
- Prove or disprove: If a polytope k-tiles with a discrete set Λ, then it also k-tiles with a **lattice** L.
- Find the number of vertices of k-tilers.
- Using the frequency spectrum $\hat{1}_P(m)$, classify all k-tiling polytopes.

One thing we keep learning is that it's very fruitful to simultaneously think about the Fourier analysis and the Discrete/Combinatorial Geometry.

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THANK YOU