Beta expansion and self-similar tilings 1

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In this first lecture, I summarize a method to construct Meyer sets and self-affine tilings by expansion in non-integer bases.
Beta expansion

Let us fix $\beta > 1$ and $\mathcal{A} = [0, \beta) \cap \mathbb{Z}$. We shall later need $\mathcal{A}^\mathbb{Z}$ the set of bi-infinite words over $\mathcal{A}$. Each element of $\mathcal{A}^\mathbb{Z}$ is written as $(a_i)_{i \in \mathbb{Z}} = \ldots a_{-1} a_0 \bullet a_1 a_2 \ldots$ where the symbol $\bullet$ is used as a usual decimal point which indicates the place where the index 1 starts.

The **beta transformation** is a piecewise linear map $T_\beta$ on $[0, 1)$ defined by

$$T_\beta : x \mapsto \beta x - \lfloor \beta x \rfloor$$
which was shown to be ergodic by Rényi [11]. Parry [10] gave the invariant measure of this system, which is absolutely continuous to the Lebesgue measure and its Radon-Nikodym derivative was made explicit. For each real $x = x_1 \in [0, 1)$, iterating beta transforms we have

$$T_\beta : x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} x_3 \xrightarrow{a_3} \ldots.$$

The label over the arrow is defined as $a_i = \lfloor \beta x_i \rfloor$. One can expand $x \in [0, 1)$ into

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{a_3}{\beta^3} + \ldots$$
and \(a_i \in \mathcal{A}\). Denote by \(d_\beta : [0, 1) \ni x \to a_1a_2\cdots \in \mathcal{A}^\mathbb{N}\). Then \(d_\beta\) is order preserving, that is, \(x < y\) implies \(d_\beta(x) <_{\text{lex}} d_\beta(y)\).

We confirm a commutative diagram:

\[
\begin{array}{ccc}
[0, 1) & \xrightarrow{T_\beta} & [0, 1) \\
\downarrow d_\beta & & \downarrow d_\beta \\
\mathcal{A}^\mathbb{N} & \xrightarrow{\sigma} & \mathcal{A}^\mathbb{N}
\end{array}
\]  

(1)

Define the realization map:

\[
\pi = \pi_\beta : a_1a_2\cdots \in \mathcal{A}^\mathbb{N} \to \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \in \mathbb{R}.
\]
Note that $\pi_\beta$ is continuous but $d_\beta$ is not. Since $\pi_\beta \circ d_\beta (x) = x$ by definition, we have $\pi_\beta (A^N) \supseteq [0, 1)$. However $A^N \not\subseteq d_\beta ([0, 1))$. If $a_1 a_2 \cdots \in A^N$ is contained in $d_\beta ([0, 1))$, we say that $a_1 a_2 \cdots \in A^N$ is **admissible**. A finite word $a_1 a_2 \ldots a_m$ of $A^\ast$ is admissible if its right completion $a_1 a_2 \ldots a_m \oplus 00 \cdots \in A^N$ is admissible. For a given positive $x$, there is an integer $m > 0$ with $\beta^{-m} x \in [0, 1)$, $x$ can be expanded like

$$
x = a_{-m} \beta^m + a_{-m+1} \beta^{m-1} + \cdots + a_0 + \frac{a_1}{\beta} + \cdots
$$

This is the **beta expansion** which is a natural generalization of usual decimal or binary expansion. By abuse of terminology,
we sometimes write

\[ d_\beta(x) = a_{-m}a_{-m+1} \ldots a_{-1}a_0 \bullet a_1a_2a_3 \ldots. \]

The expansion is **finite** if there is an \( \ell \) that \( a_n = 0 \) holds for \( n > \ell \) and we denote by

\[ x = a_{-m}a_{-m+1} \ldots a_0 \bullet a_1a_2a_3 \ldots a_\ell. \]

Set

\[ d_\beta(1 - 0) = \lim_{\varepsilon \downarrow 0} d_\beta(1 - \varepsilon). \]

by the metric of \( A^N \). Then \( d_\beta(1 - 0) \) can not be finite.
Theorem 1 ([10], [8]). A right infinite word $\omega = \omega_1\omega_2 \cdots \in A^\mathbb{N}$ is admissible if and only if $\sigma^n(\omega) <_{\text{lex}} d_\beta(1 - 0)$ holds for all $n = 0, 1, \ldots$.

This theorem shows an intimate connection between beta expansion and symbolic dynamics.
Beta expansion is long studied in relation to ergodic theory and number theory. This map has a unique invariant measure which is absolutely continuous to the 1-dimensional Lebesgue measure. The system is ergodic and has more strong properties [11, 10].

An important point of this expansion is that it has a nice way to describe it by symbolic dynamics, called beta shifts $X_\beta$.
defined by bi-infinite words \((a_i)\) over \(A\):

\[
\{(a_i) \mid a_i \in A \ \forall m, n \ \exists x \ a_i = \lfloor \beta T^{i-1}(x) \rfloor \ m \leq \forall i \leq n\}.
\]

Let \(s((a_i)) = (a_{i+1})\) be the shift operator. Then \((X_\beta, s)\) forms a topological dynamical system equipped with the local topology. By Theorem 1, we can describe \(X_\beta\) as

\[
X_\beta = \{(a_i) \mid a_i \in A \ \forall m, n \ a_m a_{m+1} \ldots a_n \ \text{is admissible}\}.
\]

Take \(\mathcal{F} \subset A^*\) and define a subset \(A_\mathcal{F}\) of \(A^\mathbb{N}\) or \(A^\mathbb{Z}\) by the infinite words whose subwords are not in \(\mathcal{F}\). Then \(A_\mathcal{F}\) is a subshift and any subshift is written in this manner. Thus \(\mathcal{F}\) is
the set of **forbidden words**. A subshift is called **of finite type** if there is a finite set $\mathcal{F}$ and it is expressed as $\mathcal{A}_\mathcal{F}$. A subshift $\mathcal{A}_\mathcal{F}$ is called **sofic** if one can choose $\mathcal{F}$ which is recognizable by a finite automaton. A subshift of finite type is sofic and the sofic shift is characterized as a factor of the shift of finite type. A sofic shift $\mathcal{A}_\mathcal{F}$ is nothing but the set of infinite labels which is generated by infinite walks on a fixed finite directed graph labelled by $\mathcal{A}$ (c.f. [9]).

The **beta shift** $X_\beta$ is a subshift of $\mathcal{A}^\mathbb{Z}$ which is defined to be a set of bi-infinite words whose all finite subwords are admissible. $X_\beta$ is sofic if and only if $d_\beta(1-0)$ is eventually periodic. Such a $\beta$ is designated as a **Parry number**. Further
$d_\beta(1-0)$ is purely periodic if and only if $A^\mathbb{N}$ is of finite type. In this case, the number $\beta$ is a simple Parry number ([10], [3]).

A Pisot number $\beta > 1$ is a real algebraic integer whose other conjugates have modulus less than one. A Salem number $\beta > 1$ is a real algebraic integer whose other conjugates have modulus not greater than one and also one of the conjugates has modulus exactly one. Denote by $\mathbb{R}_+$ the non negative real numbers.

**Theorem 2** (Bertrand [2], Schmidt [12]). If $\beta$ is a Pisot number then each element of $\mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an eventually periodic beta expansion.

Hence a Pisot number is a Parry number. In [12], a partial...
converse is shown that if all rational number in $[0, 1)$ has an eventually periodic beta expansion then $\beta$ is a Pisot or a Salem number. It is not yet known whether each element of $\mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an eventually periodic expansion if $\beta$ is a Salem number (Boyd [4], [5], [6]). See Figure 1 for a brief summary. The finiteness won’t be discussed in this lecture.
Figure 1: The classification of Parry numbers
A Parry number $\beta$ is also a real algebraic number greater than one, and other conjugates are less than $\min\{|\beta|, (1 + \sqrt{5})/2\}$ in modulus ([10], Solomyak [13]) but the converse does not hold. It is a difficult question to characterize Parry numbers among algebraic integers. ([7], [1])

Hereafter we simply say **Pisot number system** to call the method to express real numbers by beta expansion in Pisot number base. The results like [12] and [2] suggest that Pisot number system is very close to the usual decimal expansion.
Delone set and $\beta$-integers

Let $X$ be a subset of $\mathbb{R}^d$. The ball of radius $r > 0$ centered at $x$ is denoted by $B(x, r)$. A point $x$ of $X$ is isolated if there is a $\varepsilon > 0$ that $B(x, \varepsilon) \cap X = \{x\}$. The set $X$ is called discrete if each point of $X$ is isolated.

The set $X$ is uniformly discrete if there exists a positive $r > 0$ such that $B(x, r) \cap X$ is empty or $\{x\}$ for any $x \in \mathbb{R}^d$, and $X$ is relatively dense if there exists a positive $R > 0$ such that $B(x, R) \cap X \neq \emptyset$ for any $x \in \mathbb{R}^d$. A Delone set is the set in $\mathbb{R}^d$ which is uniformly discrete and relatively dense at a
time.

One can expand any positive real number $x$ by beta expansion:

$$x = a_{-m}a_{-m+1} \ldots a_0 \cdot a_1 a_2 \ldots$$

The $\beta$-integer part (resp. $\beta$-fractional part) of $x$ is defined by:

$[x]_\beta = \pi(a_{-m} \ldots a_0)$ (resp. $\langle x \rangle_\beta = \pi(a_1 a_2 \ldots)$). A real number $x$ is a $\beta$-integer if $\langle |x| \rangle_\beta = 0$. Denote by $\mathbb{Z}_\beta$ the set of $\beta$-integers and put $\mathbb{Z}_\beta^+ = \mathbb{Z}_\beta \cap \mathbb{R}_+$. 

**Proposition 1.** For any $\beta > 1$, the set of $\beta$-integers $\mathbb{Z}_\beta$ is relatively dense, discrete and closed in $\mathbb{R}$. 
From now on, we assume that $\beta$ is not an integer. As $\mathbb{Z}_\beta$ is discrete and closed, we say that $x, y \in \mathbb{Z}_\beta$ is **adjacent** if there are no $z \in \mathbb{Z}_\beta$ between $x$ and $y$.

**Proposition 2.** If $x, y \in \mathbb{Z}_\beta$ is adjacent, then there exists some nonnegative integer $n$ with $|x - y| = T^n_\beta(1)$.

The real number $\beta > 1$ is a **Delone number** if \( \{T^n_\beta(1)\}_{n=0,1,2,...} \) does not accumulates to 0. If $\beta$ is a Delone number, then $\mathbb{Z}_\beta$ is uniformly discrete with $r = \min_{n=0,1,...} T^n_\beta(1)$. With the help of Proposition 1 and 2, $\mathbb{Z}_\beta$ is a Delone set if and only if $\beta$ is a Delone number. It is plain to see that a Pisot number is a Delone number.
References


Beta expansion and self-similar tilings 2

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A tile is a compact set in $\mathbb{R}^d$ which coincides with the closure of its interior. A tiling $\mathcal{T}$ of $\mathbb{R}^d$ is a collection of tiles which covers $\mathbb{R}^d$ without interior overlaps. We assume the tiling $\mathcal{T}$ has finitely many tiles up to translations. So each tile is translationally equivalent to an element of $A = \{T_0, T_1, \ldots T_{m-1}\}$, the finite set of protiles, called alphabets.

A patch $P$ is a finite set of tiles in $\mathcal{T}$. The translation of $P$ by an element $t \in \mathbb{R}^d$ is denoted by $P + t = \{T + t \mid T \in P\}$. The diameter of $P$ is the diameter of the union of tiles in $P$. A tiling has finite local complexity (FLC) if there are only finitely many patches of a given diameter up to translation.
A tiling $\mathcal{T}$ is **repetitive** if for any patch $P$ in $\mathcal{T}$ there exists $R > 0$ that for any $x \in \mathbb{R}^d$, the intersection $B_R(x) \cap \mathcal{T}$ must contain a translation of $P$, where $B_R(x)$ is a ball of radius $R$ centered at $x$. A matrix $Q$ is called expanding if all eigenvalues of $Q$ have modulus greater than one. We assume a set equation on the alphabet

$$
QT_j = \bigcup_{i=0}^{m-1} T_i + D_{ij}
$$

for $j = 0, 1, \ldots, m - 1$, which gives rise to a **substitution rule** $\omega$ of the alphabet by inflation subdivision:

$$
\omega(T_j) = \{T_i + d_i | d_i \in D_{ij}\}
$$
The substitution rule is **primitive** if the substitution matrix \((#D_{ij})\) is primitive. This means for any \(i, j\), the tile \(T_i\) must appear in \(\omega^k(T_j)\) for some \(k\). A repetitive tiling \(\mathcal{T}\) with FLC is called **self-affine** if every patch \(P\) of \(\mathcal{T}\) is legal. The spectrum of tiling dynamical system is intimately related to the diffraction pattern generated by point sets in \(\mathbb{R}^d\), representing atomic configuration. For e.g., pure discrete spectrum of tiling dynamical system is equivalent to pure point diffraction of the associated point set explained below.

A subset \(\Lambda\) of \(\mathbb{R}^d\) is **relatively dense** if there is \(R > 0\) such that \(B_R(x) \cap \Lambda \neq \emptyset\) for any \(x \in \mathbb{R}^d\). A subset \(\Lambda\) is **uniformly discrete** if there is \(r > 0\) such that for any \(x \in \mathbb{R}^d\)
the cardinality of $B_r(x) \cap \Lambda$ is at most one. A set $\Lambda$ is a **Delone set** if $\Lambda$ is both relatively dense and uniformly discrete. Further $\Lambda$ is a **Meyer set** if $\Lambda$ and $\Lambda - \Lambda$ are Delone set. Meyer set is a well known framework to study diffraction of quasi-crystals.

We can choose for each tile $T \in \mathcal{T}$ a **reference point** which is located at the same relative position in the translationally equivalent tiles. Let $\Lambda_i$ be the reference points of an alphabet $T_i$. Then we may choose a reference points in such a way that

$$
\Lambda_i = \bigcup_{j=0}^{m-1} Q\Lambda_j + D_{ij}
$$
holds and the right hand side is disjoint. The Delone set which satisfies such a set equation is called **substitutive Delone set**. It is known that there is a nice **duality** between reference point sets and self-affine tilings which makes our study almost parallel for tilings and point sets.

- Which data $Q$ and $D_{ij}$ give rise to self-affine tilings? ([5, 6, 7, 10, 3, 12, 1, 8]),

- Under what conditions, do we see point diffraction? ([4, 11, 9, ?])

- Which tiling (point set) is pure point diffractive? ([11, 2])
Rotational beta expansion

This is a joint work with Jonathan Caalim at UP Diliman. Let $1 < \beta \in \mathbb{R}$ and $M$ be an element of the orthogonal group $O(m, \mathbb{R})$. Let $\mathcal{L}$ be a lattice of $\mathbb{R}^m$. Fix a fundamental domain $\mathcal{X}$ of $\mathcal{L}$. Then

\[
\mathbb{R}^m = \bigcup_{d \in \mathcal{L}} (\mathcal{X} + d)
\]

is a disjoint partition of $\mathbb{C}$. Define a map $T : \mathcal{X} \to \mathcal{X}$ by $T(z) = \beta M(z) - d$ where $d = d(z)$ is the unique element in $\mathcal{L}$ satisfying $\beta M(z) \in \mathcal{X} + d$. 
Given a point $z$ in $\mathcal{X}$, we obtain an expansion

$$
z = \frac{M^{-1}(d_1)}{\beta} + \frac{M^{-1}(T(z))}{\beta} = \frac{M^{-1}(d_1)}{\beta} + \frac{M^{-2}(d_2)}{\beta^2} + \frac{M^{-2}(T^2(z))}{\beta^2} = \sum_{i=1}^{\infty} \frac{M^{-i}(d_i)}{\beta^i}
$$

with $d_i = d(T^{i-1}(z))$. In this case, we write $d_T(z) = d_1d_2\ldots$. We call $T$ the rotational beta transformation and $d_T(z)$ the expansion of $z$ with respect to $T$. 
For \( m = 2, \beta > 1 \) and \( M \) is in \( SO(2, \mathbb{R}) \), the algorithm is naturally written in complex plane. Let \( \zeta \in \mathbb{C} \setminus \mathbb{R} \) with \( |\zeta| = 1 \), \( \xi, \eta_1, \eta_2 \in \mathbb{C} \) with \( \eta_1/\eta_2 \notin \mathbb{R} \). Then

\[
X = \{ \xi + x\eta_1 + y\eta_2 \mid x \in [0, 1), y \in [0, 1) \}
\]

is a fundamental domain of the lattice \( \mathcal{L} = \mathbb{Z}\eta_1 + \mathbb{Z}\eta_2 \) in \( \mathbb{C} \).

We are interested in the transform \( T(z) = \beta\zeta z - d \) and its expansion:

\[
z = \sum_{i=1}^{\infty} \frac{d_i}{\beta^i \zeta^i} \in \mathbb{C}.
\]

where \( d_i \in \mathcal{L} \).
Motivations

• What can be said on its absolutely continuous invariant measure?

• Systematic construction of self-similar tilings.

Put $\mathcal{A} := \{d(z) \mid z \in \mathcal{X}\}$. Let $\mathcal{A}^\mathbb{Z}$ (resp. $\mathcal{A}^*$) be the set of all bi-infinite (resp. finite) words over $\mathcal{A}$. We say $w \in \mathcal{A}^*$ is admissible if $w$ appears in the expansion $d_T(z)$ for some $z \in \mathcal{X} \setminus \bigcup_{n \in \mathbb{Z}} T^n(\partial(\mathcal{X}))$. Let

$$\mathcal{X}_T := \{w = (w_i) \in \mathcal{A}^\mathbb{Z} \mid w_i w_{i+1} \ldots w_j \text{ is admissible} \}.$$
The symbolic dynamical system associated to $T$ is the topological dynamics $(\mathcal{X}_T, s)$ given by the shift operator $s((w_i)) = (w_{i+1})$. We say $(\mathcal{X}_T, s)$ (or simply, $(\mathcal{X}, T)$) is sofic if there is a finite directed graph $G$ labeled by $\mathcal{A}$ such that for each $w \in \mathcal{X}_T$, there exists a bi-infinite path in $G$ labeled $w$ and vice versa.

**Lemma 1.** The system $(\mathcal{X}, T)$ is sofic if and only if $\bigcup_{n=1}^{\infty} T^n(\partial(\mathcal{X}))$ is a finite union of segments.

**A problem on the definition of soficness**

One may define **complete soficness** by considering all orbits in $\mathcal{X}$ instead of $\mathcal{X} \setminus \bigcup_{n \in \mathbb{Z}} T^n(\partial(\mathcal{X}))$. Then the results will be:
Lemma 2. The system $(\mathcal{X}, T)$ is completely sofic if and only if $(T^n(\partial(\mathcal{X})))_{n=1,2,\ldots}$ is eventually periodic as a sequence of sets.

We are not sure these definitions are the same.
So \((\mathcal{X}, T)\) to be sofic, \(\zeta\) must be a root of unity. Assume that \(\zeta\) is a \(q\)-th root of unity with \(q > 2\) and \(\xi, \eta_1, \eta_2 \in \mathbb{Q}(\zeta, \beta)\) with \(\eta_1/\eta_2 \notin \mathbb{R}\).

**Theorem 3.** Let \(\zeta\) be a \(q\)-th root of unity \((q > 2)\) and \(\beta\) be a Pisot number. Let \(\eta_1, \eta_2, \xi \in \mathbb{Q}(\zeta, \beta)\). If \(\cos(2\pi/q) \in \mathbb{Q}(\beta)\), then the system \((\mathcal{X}, T)\) is sofic.

**Corollary 4.** If \(\zeta\) is a 3rd, 4th or 6th root of unity, then the system \((\mathcal{X}, T)\) is sofic for any Pisot number \(\beta\).

**Corollary 5.** For any positive integer \(q\), there exists a Pisot number \(\beta\) which satisfies above conditions. Thus there is a self-similar tiling in \(\mathbb{R}^2\) with inflation constant \(\beta\) and \(q\)-fold rotation action, whose all tiles are polygons.
On the other hand, we can give a family of non-sofic systems when \( \zeta + \zeta^{-1} \notin \mathbb{Q}(\beta) \).

**Theorem 6.** Let \( \xi = 0 \), \( \eta_1 = 1 \) and \( \eta_2 = \zeta = \exp(2\pi \sqrt{-1}/5) \). If \( \beta > 2.90332 \) such that \( \sqrt{5} \notin \mathbb{Q}(\beta) \), then \( (\mathcal{X}, T) \) is not a sofic system.

For example, \( \beta = 3, 4, 5 \ldots \) are not sofic in this setting.
Example 7. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/3)$. Put $\beta = 1 + \sqrt{2}$. We have 9 cylinders.

Figure 1: 3-fold sofic case
Example 8. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp\left(2\pi \sqrt{-1/5}\right)$. Let $\beta = (1 + \sqrt{5})/2$. There are 40 cylinders.

Figure 2: 5-fold sofic case
Example 9. Let $\xi = 0$, $\eta_1 = 1$ and $\eta_2 = \zeta = \exp(2\pi\sqrt{-1}/7)$. Let $\beta = 1 + 2\cos(2\pi/7) \approx 2.24698$. From $r(\mathcal{L}) = 1/(2\cos(\pi/7))$, $w(\mathcal{X}) = \sin(2\pi/7)$ we have $\beta > B_1 \approx 2.00272$ and there is a unique ACIM by Theorem ??, but $\beta < B_2 \approx 2.41964$. From Theorem 3, we know that the corresponding dynamical system is sofic. Figure 3 shows the sofic dissection of $\mathcal{X}$ by 224 discontinuity segments. The number of cylinders is 3292 (!), computed by Euler’s formula.
Figure 3: Sofic 7-fold rotation
References


