1. The image above shows a “pants decomposition” of a manifold of positive genus.
(a) What is the genus?
(b) How many pants are there?
(c) How many pants are there in a generic oriented surfaces of genus g?
(d) How many cuts are needed to decompose the genus g manifold into pants?

These images show the Gauss maps about single vertices of the five Platonic polyhedra. We can label these polyhedra by a 2D Schläfli symbol \(\{n,z\}\): face of size \(n\) and vertices of degree \(z\).

2(a). Use these diagrams to determine the integral curvature per vertex - which can be worked out by recognising that the integral curvature of the entire polyhedron is \(4\pi\), since it covers the sphere once.

(b). Use Euler’s Theorem and the Gauss-Bonnet Theorem to deduce the integral curvature per vertex, expressed in terms of \(n\) and \(z\).

(c). Draw an asymmetric domain for each polyhedron and its corresponding shape on the sphere surface.
3(a) What is the integral curvature per face of the dodecahedron in (a)? Can you generalise to an arbitrary \( (n,z) \) topology?

(b) Draw the asymmetric domain on the sphere for the (5.6.6) pattern.

(c) Average appropriately the face sizes to get an expression for the Euler characteristic and integral curvature per vertex of (5.6.6).

(d) How many vertices does the (5.6.6) polyhedron have?
Answers

1. (a) There are 6 handles, the manifold is clearly orientable, so genus=6.

(b) The closed loops describe cuts that decompose the manifold into pants, with one mistake! Here is a pant decomposition - note the new cut, drawn in black.

There are, in total, 10 pants.

(c) We decompose a generic genus g manifold (oriented) into 2 “ends” and (g-2) “internal handles” (see image below). The ends contribute 2 pants in total, the internal handles, 2 each, so the total number is 2+2(g-2) = 2(g-1).
(d). Each pant has 3 boundary seams, formed by the cuts. Each seam is a 1/2 cut loop, since a cut dissects the manifold into a pair of boundary seams. From (c), the total number of seams is 3.2(g-1), so the total number of cuts is 3(g-1).

2. (a) All of the polyhedra are Platonic, so the integral curvature per vertex is the total area of the Gauss map for the polyhedron, divided by the number of vertices, $4\pi / N$. ($N=4, 6, 8, 12, 20$).

(b) Euler’s Theorem can be used to get an effective Euler’s characteristic per vertex:

$$\chi = V - E + F.$$  

so the Euler characteristic per vertex is:

$$\frac{\chi}{V} = 1 - \frac{E}{V} + \frac{F}{V} = 1 - \frac{z}{2} + \frac{z}{n}.$$  

Gauss-Bonnet tells us that $$\int \int_{surface} Kd\sigma = 2\pi \chi,$$ so the integral curvature per vertex is

$$\frac{\pi}{n}\left\{4 - (n - 2)(z - 2)\right\}.$$
The asymmetric domains are tetrahedra in 3d space, whose faces intersect the surrounding spheres to form the yellow arcs, forming triangular asymmetric domains on the sphere. All of the asymmetric domains are bounded by mirrors: planes in 3d euclidean space and along sphere geodesics on the 2d sphere.

3 (a) Two-dimensional “duals” are those that swap faces for vertices and vertices for faces. That is reflected in the topology by \( \{n,z\} \rightarrow \{z,n\} \). So, for example, the tetrahedron \( \{3,3\} \) is self dual, the cube \( \{4,3\} \) is dual to the octahedron \( \{3,4\} \) and the dodecahedron \( \{5,3\} \) and icosahedron \( \{3,5\} \) are duals. We can use the result of 2(b), replacing faces for vertices, to get the curvature per face:

\[
\frac{\pi}{z} \left\{ 4 - (n - 2)(z - 2) \right\}
\]
(c) Each (5.6.6) vertex “sees” $1/5$th of a pentagonal face, and two times $1/6$ of hexagonal faces. So the average ring size is

\[
\bar{n} = \frac{\frac{1}{5}(5) + \frac{2}{6}(6)}{\frac{1}{5} + \frac{2}{6}} = \frac{45}{8}
\]

(d) From the answers to 3(a) and 3(c), and substituting the average ring size for $n$ in 3(a), we get an integral curvature per vertex of $\pi/15$. Since the complete polyhedron has integral curvature $4\pi$, there are 60 vertices. This polyhedron describes the famous "buckminsterfullerene" form of graphitic carbon C60.