Two-dimensional Crystallography

via topology and orbifolds
Many 3D structures are also 2D non-euclidean patterns!
• manifolds and Euler characteristic
• construction of manifolds from caps, pants and cross caps
• Gaussian curvature of manifolds
• manifolds as orbifolds: wallpaper
<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td><strong>faces</strong></td>
<td><strong>4</strong></td>
</tr>
<tr>
<td><strong>-edges</strong></td>
<td><strong>6</strong></td>
</tr>
<tr>
<td><strong>vertices</strong></td>
<td><strong>4</strong></td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td><strong>2</strong></td>
</tr>
</tbody>
</table>
Faces - Edges + Vertices = 6 - 12 + 8 = 2
Faces - Edges + Vertices = 12 - 24 + 14 = 2
The face, edge, vertex sum depends on **topology** only:

\[ F - E + V = \chi \]
Face, edge, vertex count is unchanged if we blow the polyhedron onto a sphere...

Euler’s theorem for convex polyhedra:
Any map on the sphere divides it into \((F)\) faces, \((E)\) edges and \((V)\) vertices, where:

\[ F - E + V = 2 \]
“toroidal polyhedra”

Faces - Edges + Vertices
32 - 64 + 32 = 0
"toroidal polyhedra"

Faces - Edges + Vertices = 0

http://upload.wikimedia.org/wikipedia/commons/8/84/Toroidal_polyhedron.gif
Surface topology sets the Euler characteristic, $\chi$.

- Sphere: $\chi = 2$
- Torus: $\chi = 0$
- Bitorus: $\chi = -2$
- Tritorus: $\chi = -4$
“infinite polyhedra”

\[ \chi = -\infty \]
Per cubic unit cell:
<table>
<thead>
<tr>
<th>cubic unit cell:</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>edges</td>
<td>12</td>
</tr>
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<td>vertices</td>
<td>8</td>
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<tr>
<td>Sum</td>
<td>-4</td>
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</table>
\[ \chi = -4 \]
\[ \chi = -4 \]
Pants decomposition for a higher-gneus manifold:
Build all ‘nice’ surfaces from pairs of pants:
Euler characteristic of pants:

\[ V = 12/2 = 6 \; ; \; E = 6 + 6/2 = 9; \; F = 2 \]

\[ X_{\text{pants}} = -1 \]
.....and caps:

\[ V = 2 \ ; \ E = 3 ; \ F = 2 \]

\[ \chi_{\text{cap}} = 1 \]
Euler characteristic of ‘nice’ surfaces:

\[ \chi = \# \text{ pants} \cdot \chi_{\text{pants}} + \# \text{ caps} \cdot \chi_{\text{cap}} \]
2 pants, 2 caps

\[ \chi = 0 \]
3 pants, 3 caps

\[ \chi = 0 \]
GENUS of ‘nice’ surfaces, “\( g \):

\[
\chi = (\# \text{ caps} - \# \text{ pants})
\]

\[
g = 1 - \frac{\chi}{2}
\]

\[
g = 1 + \frac{\# \text{ pants} - \# \text{ caps}}{2}
\]
Surface topology sets the Euler characteristic, $\chi$ and genus.

- $\chi = 2$, genus = 0
- $\chi = 0$, genus = 1
- $\chi = -2$, genus = 2
- $\chi = -4$, genus = 3
Euler characteristic describes surface topology.

\[ F - E + V = \chi \]

Genus describes number of loops or “handles”

\[ g = 1 - \frac{\chi}{2} \]
8 pants, 6 caps: $\chi = -2$, genus=2
‘nice’ surfaces with punctures:

\[ \chi = \# \text{pants} \cdot \chi_{\text{pants}} + \# \text{caps} \cdot \chi_{\text{cap}} + \# \text{holes} \cdot \chi_{\text{holes}} \]

\[ = \# \text{caps} - \# \text{pants} - \# \text{holes} \]
- orientable (nice) vs. non-orientable (nasty) surfaces
Möbius strip

http://www.cs.technion.ac.il/~gershon/EscherForReal/MoebiusAnt.gif
What a difference a $\pi$-twist makes....

single puncture!  

double puncture!
What are nasty surfaces?
“nasty” surfaces are non-orientable.

Orientable and non-orientable surfaces are different topological species.
Moebius band

1 Face
3 Edges
2 Vertices

\[ F - E + V = 0 \]
1 Face
3 Edges
2 Vertices

\[ F - E + V = 0 \quad \chi = 0 \]

Nasty surfaces can have same Euler characteristic as nice surfaces, but topologically different
(vertices and edges are shared with adjacent modules):

\[ V = \frac{2}{2} = 1 \ ; \ E = \frac{2}{2} + 1 = 2 \ ; \ F = 1 \]

\[ \chi_{xcap} = 0 \]
Moebius band is topologically identical to cross-cap!

\[ \chi_{xxcap} = 0 \]
another nasty surface: Klein "bottle"
1 Face
2 Edges
1 Vertex

\[ F - E + V = 0 \]

\[ \chi = 0 \]
Non-oriented genus = 1

$\chi = 0$
A half-Klein bottle is a Möbius band

Zip two Möbius bands together along boundary: form a Klein bottle!
Non-orientable surfaces.....

Cannot be built from pants and caps, instead, use pants, caps and cross-caps
replace pants by “handle”:

$$\text{handle} = 1 \text{ pant} + 1 \text{ cap}$$

ANY nice surface = sphere + handles (+ boundary)

sphere + 3 handles (genus 3)
To build any nice surface:

1. Start with a sphere (2 caps)
To build any nice surface:

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2. Add handles (2 punctures, 1 pant + 1 cap)

\[ \chi_{\text{handle}} = 2\chi_{\text{puncture}} + \chi_{\text{pant}} + \chi_{\text{cap}} = -2 - 1 + 1 = -2 \]
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3. Add boundary punctures

\[ \chi_{\text{boundary}} = -1 \]
ANY nasty surface = sphere + handles + xcaps (+ boundary)

e.g. Boys surface
    = sphere + xcap
To build any surface:

1. Start with a sphere (2 caps)
   \[ \chi_{\text{sphere}} = 2 \]

2. Add handles (2 punctures, 1 pant + 1 cap)
   \[ \chi_{\text{handle}} = 2\chi_{\text{puncture}} + \chi_{\text{pant}} + \chi_{\text{cap}} = -2 - 1 + 1 = -2 \]

4. Add boundary punctures
   \[ \chi_{\text{boundary}} = -1 \]
ANY surface = sphere + handles + xcaps + boundary!

\[ \chi = 2 - 2(\#handles) - (\#xcaps) - (\#boundaries) \]
# Conway symbols

<table>
<thead>
<tr>
<th>modules</th>
<th>symbol</th>
<th>Diagram</th>
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</thead>
<tbody>
<tr>
<td>handle</td>
<td>O</td>
<td><img src="image" alt="handle" /></td>
</tr>
<tr>
<td>cross-cap</td>
<td>X</td>
<td><img src="image" alt="cross-cap" /></td>
</tr>
<tr>
<td>boundary</td>
<td>*</td>
<td>remove a <img src="image" alt="boundary" /></td>
</tr>
</tbody>
</table>
\[ \chi = 2 - 2(\#\text{handles}) - (\#\text{xcaps}) - (\#\text{boundaries}) \]

Notice that surface features (handles, boundaries, xcaps) induce negative \( \chi \).

**Conway symbol** describes the surface

\[ o = \text{handle} \]

\[ * = \text{boundary loop} \]

\[ x = \text{crosscap} \]

**Example:** \( O^{***}XX \)

\[ \chi(O^{***}XX) = 2 - [2(1) + 1(3) + 1(2)] = -5 \]

better \( ***XXXX \)
Summarising:

Surface topology is independent of shape details

Quantify topology by

- *nice*: pants, caps and holes only
  ★ 2-sided, orientable

- *nasty*: pants, caps, holes & crosscaps
  ★ 1-sided, non-orientable

Characterise topology by value of $\chi$

Characterise geometry by sign of $\chi$
Topology ($\chi$) is related to surface curvature ($K$)
Gauss-Bonnet Theorem: geometry from topology

\[ \int \int K \, d\alpha = 2\pi \chi \]

- Surface integral of Gaussian curvature
- Euler characteristic
\[ \int \int K \, da \quad = \quad \text{solid angle traced out by normals to surface} \]

Figure 1.18. Planar vs. solid angle construction. A planar angle $\theta$ is equal to the perimeter of a circular arc of radius one swept out by a radial edge. The solid angle is the area of the region on the unit sphere traced out by a radial edge that sweeps through the entire solid angle. The vertex angles of the resulting spherical polygon (in this case, a triangle) are equal to the dihedral angles between adjoining faces, $\beta_i$.

(include sign of solid angle: “Gauss map”)
Area of pole region = integral (Gauss) curvature

Figure 1.22. The pole region on the sphere due to a \( \{n, z\} \) polyhedral vertex. The region is a face of a spherical polyhedron (cf. Figure 1.20, whose vertices are the pole figures of all the \( z \) faces that contain that vertex.

Area of pole region of complete polyhedron = \( 4\pi \)

\( \chi = 2 \)
\[ \int \int K \, da = 2\pi \chi \]

\[ \langle K \rangle = \frac{2\pi \chi}{\text{Area}} \]

Assume intrinsic homogeneity: Constant $K$
(i.e. no curvature variations)

\[ \int \int K \, da = K \cdot \text{Area} \]
... a torus has one handle

\[ \chi = 2 - 2(#\text{handles}) - (#\text{xcaps}) - (#\text{boundaries}) \]

\[ \chi = 0 \]

Conway symbol: \( o \)
... a torus has $\chi = 0$ i.e. it is - on average - flat!

exactly equal contributions of + and - Gaussian curvature

Gauss curvature

positive $\rightarrow 0$ $\rightarrow$ negative
a tritorus has three handles.....

\[ \chi = 2 - 2(\#\text{handles}) - (\#\text{xcaps}) - (\#\text{boundaries}) \]

\[ \chi = -4 \]

\[ <K> = \frac{2\pi \chi}{\text{Area}} \]

... so a tritorus is hyperbolic, with negative <K>
... a genus-3 tritorus is -- on average -- HYPERBOLIC

Conway symbol: ooo
... an infinite genus 3-periodic surface is -- on average -- HYPERBOLIC.

Conway symbol:

00......

Gauss curvature

0 → negative
$\langle K \rangle > 0$  elliptic
$\langle K \rangle = 0$  euclidean
$\langle K \rangle < 0$  hyperbolic
local homogeneous 2D flat geometry can be globally extended in 3D space:

euclidean plane = normal plane
local homogeneous 2D **elliptic** geometry can be globally extended in 3D space:

elliptic plane = sphere
2D hyperbolic space is much “bigger” than 2D flat space

e.g. hyperbolic: area of a disc $\sim$ exponential(radius)

flat: area of a disc $\sim$(radius) $^2$
We represent the hyperbolic plane by the Poincaré model

\[ ds = \frac{2|dz|}{1 - |z|^2} \]
what are these manifolds?

cylinder

Moebius strip

torus

Klein bottle
which ones are nice (oriented), nasty?

cylinder

torus

Moebius strip

Klein bottle
how many caps? xcaps? handles? boundaries?

cylinder

Moebius strip

torus

Klein bottle
what are their Conway symbols?

cylinder

Moebius strip

torus

Klein bottle
what is their geometry?

cylinder, Moebius strip

euclidean, $K=0$

torus, Klein bottle
What is their geometry?

- Cylinder
- Möbius strip
- Torus
- Klein bottle
Let’s build the universal cover of these manifolds...

The universal covers tile the Euclidean plane, $E^2$.

torus

Klein bottle
build the “universal cover” of o:
O = p1 wallpaper!

torus
Klein bottle \( xx = \text{pg wallpaper} \)
cylinder

Moebius strip

torus

Klein bottle
** = pm wallpaper!
Conway symbols describe “orbifolds”

Orbifolds describe symmetric patterns