International School on Fundamental Crystallography

and

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Overview.

General introduction to representation theory

- basic concepts of group representations
- equivalence of representations
- invariant subspaces and reducibility
- characters and character tables
- theorems of orthogonality
- decomposition into irreps

Group representations

Definition

- A (complex) group representation of a group $\mathcal{G}$ is a homomorphism
$$\mathbf{D} : \mathcal{G} \to \mathrm{GL}_n(\mathbb{C}),$$
i.e. a mapping from $\mathcal{G}$ into the invertible $n \times n$-matrices over $\mathbb{C}$ which is compatibel with the group operations:
$$\mathbf{D}(gh) = \mathbf{D}(g)\,\mathbf{D}(h).$$
- The size n of the matrices, is called the degree or dimension of $\mathbf{D}$, denoted by $\deg(\mathbf{D})$ or $\dim(\mathbf{D})$.
- Via such a representation, $\mathcal{G}$ acts on the vector space $\mathbb{C}^n$ by the usual product of a matrix with a vector: $g(\mathbf{v}) = \mathbf{D}(g) \cdot \mathbf{v}$.

First examples

- A cyclic group of order n generated by an element g has a 1-dimensional representation **D** given by

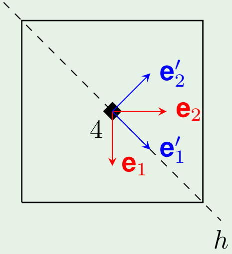
$$\mathbf{D}(g) = \left(e^{\frac{2\pi i}{n}}\right), \text{ thus } g \text{ maps } \mathbf{v} \in \mathbb{C}^1 \text{ to its multiple } e^{\frac{2\pi i}{n}}\mathbf{v}.$$

- The group $SO_2(\mathbb{R})$ of rotations on $\mathbb{R}^2$ has a representation **D** of degree 2 which maps a rotation $\mathbf{r}_\theta$ by the angle θ to

$$\mathbf{D}(\mathbf{r}_\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

- Every group $\mathcal{G}$ has a representation of degree 1, called the trivial representation or identity representation of G, which maps every group element g to the matrix (1).
  The trivial representation fixes every vector $\mathbf{v} \in \mathbb{C}^1$.

Example



The symmetry group $\mathcal{G} = 4mm$ of a square is generated by a 4-fold rotation g and a diagonal reflection h in the line $x = y$.

- With respect to the basis $\mathbf{e}_1, \mathbf{e}_2$, the action of $\mathcal{G}$ on the plane gives rise to the representation **D** with

$$\mathbf{D}(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- With respect to the different basis $\mathbf{e}_1', \mathbf{e}_2'$, we obtain a different representation **D**′ with

$$\mathbf{D}'(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D}'(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Vector representation

If a group $\mathcal{G}$ is a group of $n \times n$ matrices, e.g. a point group in 2D or 3D, then the identical mapping $g \mapsto g$ is a representation, called the vector representation or natural representation of $\mathcal{G}$.

Quick quiz

In the symmetry group $\mathcal{G} = 2mm = \{1, 2, m_{10}, m_{01}\}$ of a rectangle, the elements $2, m_{10}, m_{01}$ all have order 2. The matrix $\begin{pmatrix}-1\end{pmatrix}$ also has order 2. Is

$$\mathbf{D} : 1 \mapsto \begin{pmatrix}1\end{pmatrix}, \quad 2 \mapsto \begin{pmatrix}-1\end{pmatrix}, \quad m_{10} \mapsto \begin{pmatrix}-1\end{pmatrix}, \quad m_{01} \mapsto \begin{pmatrix}-1\end{pmatrix}$$

a representation of $\mathcal{G}$?

Answer

No, $\mathbf{D}(2)\,\mathbf{D}(m_{10}) = \begin{pmatrix}1\end{pmatrix}$,
but $\mathbf{D}(2 \cdot m_{10}) = \mathbf{D}(m_{01}) = \begin{pmatrix}-1\end{pmatrix}$ is different.

Exercise

A group $\mathcal{G}$ has a representation **D** such that for two elements g, h of $\mathcal{G}$ one has

$$\mathbf{D}(g) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{D}(h) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Determine $\mathbf{D}(gh)$, $\mathbf{D}(g^2)$, $\mathbf{D}(h^2)$ and $\mathbf{D}(hg)$.
Is $\mathcal{G}$ an abelian group?

Answer

$$\mathbf{D}(gh) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \; \mathbf{D}(g^2) = \mathbf{D}(h^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \; \mathbf{D}(hg) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

The group is not abelian, since $gh = hg$ would imply $\mathbf{D}(gh) = \mathbf{D}(hg)$, which is not the case.

Note: The opposite conclusion does not hold, one may have $\mathbf{D}(gh) = \mathbf{D}(hg)$ without the group being abelian (think of the trivial representation).

Isomorphic groups

- We call two groups $\mathcal{G}$ and $\mathcal{G}'$ isomorphic if there is a bijective homomorphism φ between $\mathcal{G}$ and $\mathcal{G}'$, i.e. a mapping φ which:
  - assigns every element of $\mathcal{G}'$ to precisely one element of $\mathcal{G}$, which means that $\mathcal{G}$ and $\mathcal{G}'$ have the same number of elements (if finite);
  - is compatible with the group operations, i.e. $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$.

  Such a mapping is called an isomorphism between $\mathcal{G}$ and $\mathcal{G}'$.
- An isomorphism can be regarded as a relabelling of the group elements, the multiplication tables of $\mathcal{G}$ and $\mathcal{G}'$ have precisely the same structure if we replace each $g_i \in \mathcal{G}$ by its image $\varphi(g_i) \in \mathcal{G}'$.
- The 2D point group $4mm$ and the 3D point groups $4mm$ and 422 are all isomorphic (to the abstract group $\mathcal{D}_4$):

$4mm$ (2D)	1	2	4^+	4^-	m_{10}	m_{01}	m_{11}	$m_{1\bar{1}}$
$4mm$ (3D)	1	2_{001}	4^+_{001}	4^-_{001}	m_{100}	m_{010}	m_{110}	$m_{1\bar{1}0}$
422 (3D)	1	2_{001}	4^+_{001}	4^-_{001}	2_{010}	2_{100}	$2_{1\bar{1}0}$	2_{110}

- Representations only depend on the isomorphism type of $\mathcal{G}$, if $\mathbf{D}'$ is a representation of $\mathcal{G}'$ and φ is an isomorphism from $\mathcal{G}$ to $\mathcal{G}'$, then $\mathbf{D}(g) = \mathbf{D}'(\varphi(g))$ is a representation of $\mathcal{G}$.

Kernel of a representation

Definition

- The subgroup $\ker \mathbf{D} = \{g \in \mathcal{G} \mid \mathbf{D}(g) = \mathbf{I}_n\}$ of elements of $\mathcal{G}$ mapped to the $n \times n$ identity matrix is called the kernel of the representation.
- The kernel of a representation is a normal subgroup of $\mathcal{G}$.

Significance of the kernel $\mathcal{H} = \ker \mathbf{D}$

- The kernel $\mathcal{H} = \ker \mathbf{D}$ consists of those elements $g \in \mathcal{G}$ which fix all vectors in $\mathbb{C}^n$, since $\mathbf{D}(g)\mathbf{v} = \mathbf{I}_n\mathbf{v} = \mathbf{v}$.
- A representation $\overline{\mathbf{D}}'$ of the factor group $\mathcal{G}/\mathcal{H}$ can be regarded as a representation $\mathbf{D}'$ of $\mathcal{G}$ having $\mathcal{H}$ in its kernel, by setting $\mathbf{D}'(g) = \overline{\mathbf{D}}'(g\mathcal{H})$ (but $\ker \mathbf{D}'$ may be larger than $\mathcal{H}$).
- All representations of $\mathcal{G}$ having $\mathcal{H}$ in their kernel occur in this way.

Equivalence of representations

Definition

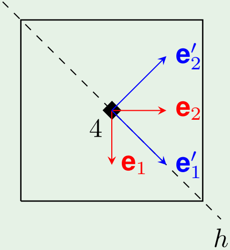
Two representations **D** and **D**$'$ of a group $\mathcal{G}$ are called equivalent if there exists an invertible $n \times n$ matrix $\mathbf{X}$ such that

$$\mathbf{D}'(g) = \mathbf{X}^{-1}\mathbf{D}(g)\mathbf{X}$$

Why should we call these representations equivalent?

- Let the matrices of **D** represent the action of $\mathcal{G}$ on $\mathbb{C}^n$ with respect to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathbb{C}^n$.
- Think of $\mathbf{X}$ as a basis transformation from $\mathbf{v}_1, \ldots, \mathbf{v}_n$ to a new basis $\mathbf{v}'_1, \ldots, \mathbf{v}'_n$ of $\mathbb{C}^n$.
- Then **D**$'$ expresses the same action of $\mathcal{G}$ on $\mathbb{C}^n$ with respect to the new basis $\mathbf{v}'_1, \ldots, \mathbf{v}'_n$.

Example



The representations **D** and **D**′ of $\mathcal{G} = 4mm$ with respect to the bases $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}'_1, \mathbf{e}'_2$ are

$$\mathbf{D}(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and}$$

$$\mathbf{D}'(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D}'(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The basis transformation from $\mathbf{e}_1, \mathbf{e}_2$ to $\mathbf{e}'_1, \mathbf{e}'_2$ is

$$\mathbf{X} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

- One checks that

$$\mathbf{D}(g)\mathbf{X} = \mathbf{X}\mathbf{D}'(g) \text{ and } \mathbf{D}(h)\mathbf{X} = \mathbf{X}\mathbf{D}'(h)$$

which is equivalent to

$$\mathbf{D}'(g) = \mathbf{X}^{-1}\mathbf{D}(g)\mathbf{X} \text{ and } \mathbf{D}'(h) = \mathbf{X}^{-1}\mathbf{D}(h)\mathbf{X}.$$

Exercise

Let $\mathcal{C}_4$ be a cyclic group of order 4 generated by the element g.
Two of the following three representations of $\mathcal{C}_4$ are equivalent:

$$\mathbf{D}(g) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{D}'(g) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{D}''(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Determine which two are equivalent, find a conjugating matrix and give an argument why the remaining one is not equivalent.

Hints:

- Look at the orders of the matrices, conjugate matrices have the same order.
- Finding $\mathbf{X}$ such that $\mathbf{D}'(g) = \mathbf{X}^{-1}\mathbf{D}(g)\mathbf{X}$ is equivalent to finding $\mathbf{X}$ such that $\mathbf{X}\mathbf{D}'(g) = \mathbf{D}(g)\mathbf{X}$, but the latter is easier to solve.

Answer

Conjugating by a basis transformation does not change the order of a matrix, $\mathbf{D}(g)$ and $\mathbf{D}''(g)$ have order 4, but $\mathbf{D}'(g)$ has order 2, thus $\mathbf{D}'$ can not be equivalent to any of the other two representations.

Evaluating $\mathbf{X}\mathbf{D}''(g) = \mathbf{D}(g)\mathbf{X}$ for $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ gives $\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = \begin{pmatrix} ia & ib \\ -ic & -id \end{pmatrix}$ and choosing $a = c = 1$ gives the solution

$$\mathbf{X} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

(other choices give other solutions, but check that $\det \mathbf{X} \neq 0$).

Unitary representations

Definition

- A complex $n \times n$-matrix is called a unitary matrix if its columns form an orthonormal basis of $\mathbb{C}^n$ (with respect to the standard scalar product $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i {w_i}^*$ on $\mathbb{C}^n$), i.e. if $\mathbf{A}^\mathsf{T} \, \mathbf{A}^* = \mathbf{I}_n$.
- The inverse of a unitary matrix is the transposed of the complex conjugate matrix, i.e. $\mathbf{A}^{-1} = (\mathbf{A}^*)^\mathsf{T} = \mathbf{A}^\dagger$.
  This matrix $\mathbf{A}^\dagger = (\mathbf{A}^*)^\mathsf{T} = (\mathbf{A}^\mathsf{T})^*$ is called the Hermitian conjugate of $\mathbf{A}$.
- A unitary matrix having only real entries is an orthogonal matrix.
- A representation **D** such that every $\mathbf{D}(g)$ is a unitary matrix is called a unitary representation.

Theorem

Every representation **D** of a finite group $\mathcal{G}$ is equivalent to a unitary representation.

Example

- The representation

$$\mathbf{D}(g) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \ \mathbf{D}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  expresses the action of $\mathcal{G} = 3m$ with respect to the conventional basis of a hexagonal lattice.
- It is equivalent to the unitary representation

$$\mathbf{D}'(g) = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad \mathbf{D}'(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  which is written with respect to an orthonormal basis of $\mathbb{C}^2$.
- It is also equivalent to the unitary representation

$$\mathbf{D}''(g) = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \ \mathbf{D}''(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  which is written with respect to an orthonormal basis of $\mathbb{R}^2$.

Direct sums of representations

Definition

Let $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ be representations of degrees n_1 and n_2, respectively. Joining the representations $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ as diagonal blocks into matrices of size $n_1 + n_2$ gives a representation

$$\mathbf{D}^{(1)} \oplus \mathbf{D}^{(2)}(g) = \left(\begin{array}{c|c} \mathbf{D}^{(1)}(g) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}^{(2)}(g) \end{array}\right)$$

of degree $n_1 + n_2$ which is called the direct sum of $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$.

Remark

The direct sum construction shows, that even up to equivalence there are infinitely many different representations of a group $\mathcal{G}$, since it allows to construct representations of arbitrary large degree.

Example

The representation

$$\mathbf{D}^{(1)} \oplus \mathbf{D}^{(2)}(g) = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right), \quad \mathbf{D}^{(1)} \oplus \mathbf{D}^{(2)}(h) = \left(\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

is the direct sum of the 1-dimensional representation

$$\mathbf{D}^{(1)}(g) = \big(1\big), \quad \mathbf{D}^{(1)}(h) = \big(-1\big)$$

and the 2-dimensional representation

$$\mathbf{D}^{(2)}(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{D}^{(2)}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of the symmetry group $\mathcal{G} = 4mm$ of a square.

Reducible representations

Definition

- A representation **D** is called reducible if it is equivalent to one of the form

$$\left(\begin{array}{c|c} \mathbf{D}^{(1)}(g) & \mathbf{H}^{(12)}(g) \\ \hline \mathbf{0} & \mathbf{D}^{(2)}(g) \end{array}\right).$$

  In this case, $\mathbb{C}^n$ has a $\mathcal{G}$-invariant subspace $\mathbb{V}$ different from $\{\mathbf{0}\}$ and $\mathbb{C}^n$ which is closed under the action of $\mathcal{G}$ and on which $\mathcal{G}$ acts by $\mathbf{D}^{(1)}$.

- If **D** is even equivalent to a representation of the form

$$\left(\begin{array}{c|c} \mathbf{D}^{(1)}(g) & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{D}^{(2)}(g) \end{array}\right)$$

  then **D** is called decomposable or fully reducible.
  In that case, $\mathbb{C}^n$ has a further $\mathcal{G}$-invariant subspace $\mathbb{W}$, called a complement of $\mathbb{V}$, on which $\mathcal{G}$ acts by $\mathbf{D}^{(2)}$ and such that $\mathbb{C}^n = \mathbb{V} \oplus \mathbb{W}$.

Irreducible representations

Theorem

Every complex representations of a finite group is fully reducible.

A complement of a $\mathcal{G}$-invariant subspace $\mathbb{V}$ can be constructed explicitly: for a unitary representation, the orthogonal complement

$$\mathbb{W} = \mathbb{V}^{\perp} = \{\mathbf{w} \in \mathbb{C}^n \mid \mathbf{v} \cdot \mathbf{w} = 0 \text{ for all } \mathbf{v} \in \mathbb{V}\}$$

is also a $\mathcal{G}$-invariant subspace.

Definition

If a representation **D** is not reducible, i.e. if the only $\mathcal{G}$-invariant subspaces of $\mathbb{C}^n$ are the trivial subspaces $\{\mathbf{0}\}$ and $\mathbb{C}^n$, the representation **D** is called irreducible.

Examples

- The representation **D** of $\mathcal{C}_2 = \{e, g\}$ with $\mathbf{D}(g) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is reducible because it has two 1-dimensional invariant subspaces $\mathbb{V}$ and $\mathbb{W}$ spanned by the eigenvectors $\mathbf{b}_1 = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{b}_2 = \mathbf{e}_1 - \mathbf{e}_2$ of $\mathbf{D}(g)$.
  Transforming **D** to the basis $\mathbf{b}_1, \mathbf{b}_2$ gives the equivalent representation
  $$\mathbf{D}'(g) = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -1 \end{array}\right).$$
- The representation $\mathbf{D}(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{D}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $\mathcal{G} = 4mm$ is irreducible, because a non-trivial invariant subspace must be 1-dimensional, i.e. spanned by a common eigenvector of the two matrices (possibly for different eigenvalues).
  The eigenvectors of $\mathbf{D}(h)$ are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, but none of these vectors is an eigenvector of $\mathbf{D}(g)$.

Exercise

Show that the representation **D** of the cyclic group $\mathcal{G} = \mathcal{C}_4 = \langle g \rangle$ of order 4 given by $\mathbf{D}(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is reducible.

Give explicitly the bases of the $\mathcal{G}$-invariant subspaces.

Hint: The eigenvalues of $\mathbf{D}(g)$ are i and $-i$.

Answer

Eigenvector for the eigenvalue i:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} ix \\ iy \end{pmatrix} \Rightarrow y = -ix, \text{ e.g. } \mathbf{b}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Eigenvector for the eigenvalue $-i$:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} -ix \\ -iy \end{pmatrix} \Rightarrow y = ix, \text{ e.g. } \mathbf{b}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

Basis transformation $\mathbf{X} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ has inverse $\mathbf{X}^{-1} = \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$

and $\mathbf{X}^{-1}\mathbf{D}(g)\mathbf{X} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Tutorial

Show that the representation

$$\mathbf{D}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{D}(h) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

of $\mathcal{G} = \mathcal{D}_3$ is reducible.

- Decompose **D** into a direct sum of irreducible representations.
- Give explicitly the bases of the $\mathcal{G}$-invariant subspaces.

Hint: Find a common fixed vector of $\mathbf{D}(g)$ and $\mathbf{D}(h)$ and show that the action on the orthogonal complement is irreducible.

Tutorial

- A common eigenvector for the eigenvalue 1 is $\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, the vectors $\mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ span the orthogonal complement.
- Conjugating by the matrix $\mathbf{X} = \begin{pmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ transforms the matrices $\mathbf{D}(g)$ and $\mathbf{D}(h)$ into block diagonal form. However, having found the bases of the subspaces already, the representations on the subspaces can be derived directly.
- The representation $\mathbf{D}^{(1)}$ of $\mathcal{G}$ on $\mathbb{V} = \langle \mathbf{b}_1 \rangle$ is

$$\mathbf{D}^{(1)}(g) = \begin{pmatrix} 1 \end{pmatrix}, \quad \mathbf{D}^{(1)}(h) = \begin{pmatrix} 1 \end{pmatrix}$$

since b_1 is fixed both by $\mathbf{D}(g)$ and $\mathbf{D}(h)$.

Tutorial (ctd.)

- For the representation $\mathbf{D}^{(2)}$ on $\mathbb{W} = \langle \mathbf{b}_2, \mathbf{b}_3 \rangle$ compute

  $$\mathbf{D}(g)\mathbf{b}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \mathbf{b}_3 \text{ and}$$

  $$\mathbf{D}(g)\mathbf{b}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = -\mathbf{b}_2 - \mathbf{b}_3.$$

  Similarly, one gets $\mathbf{D}(h)\mathbf{b}_2 = -\mathbf{b}_2$ and $\mathbf{D}(h)\mathbf{b}_3 = \mathbf{b}_2 + \mathbf{b}_3$, thus

  $$\mathbf{D}^{(2)}(g) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{D}^{(2)}(h) = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

- Since $\mathbf{D}^{(2)}(h)$ is of order 2, the eigenvalues are 1 or -1. An eigenvector with eigenvalue 1 is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and an eigenvector with eigenvalue -1 is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, but none of these is an eigenvector for $\mathbf{D}^{(2)}(g)$, hence $\mathbf{D}^{(2)}$ is irreducible.

Finding the transformation to a known subrepresentation

- Suppose we have a reducible representation $\mathbf{D}$ of degree n of $\mathcal{G}$ and know that it contains a representation $\mathbf{D}'$ of degree m on a $\mathcal{G}$-invariant subspace $\mathbb{V}$.
- A basis for $\mathbb{V}$ with respect to which $\mathcal{G}$ acts by $\mathbf{D}'$ is given by the columns of an $n \times m$-matrix $\mathbf{X}$ for which $\mathbf{D}(g)\mathbf{X} = \mathbf{X}\mathbf{D}'(g)$ for all $g \in \mathcal{G}$ (generators are sufficient).
- If $\mathbf{D}'$ is 1-dimensional, this is just the equation for an eigenvector.
- If in the previous example we would have known that the representation $\mathbf{D}^{(2)}$ is contained, we could have computed its basis $\mathbf{b}_2, \mathbf{b}_3$ by solving the system of equations obtained from

$$\mathbf{D}(g)\mathbf{X} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \mathbf{X}\mathbf{D}^{(2)}(g)$$

$$\mathbf{D}(h)\mathbf{X} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \mathbf{X}\mathbf{D}^{(2)}(h)$$

Irreps

Theorem

A finite group $\mathcal{G}$ has up to equivalence only a finite number of irreducible representations.

Terminology

- The non-equivalent irreducible representations of a group $\mathcal{G}$ are called the irreps of $\mathcal{G}$.
- From now on, we will suppress the term 'up to equivalence'. With different irreps we always mean non-equivalent irreps.

Some facts on irreps and their degrees

- The number of different irreps of $\mathcal{G}$ is equal to the number of conjugacy classes of $\mathcal{G}$.
- Let $n_1, \ldots, n_r$ be the degrees of the different irreps of $\mathcal{G}$. Then the sum of the squares of these degrees is equal to the group order, i.e.
$$|\mathcal{G}| = n_1^2 + n_2^2 + \ldots + n_r^2.$$
- In particular, the irreps of an abelian group all have degree 1, since every element of $\mathcal{G}$ forms a conjugacy class on its own.
- The degree of an irrep of $\mathcal{G}$ divides the group order $|\mathcal{G}|$.
- The number of irreps of degree 1 divides the group order. It is equal to the order of the largest abelian factor group of $\mathcal{G}$ (called the commutator factor group).

Example

The rotation group $\mathcal{O}$ of the cube has order 24 and 5 conjugacy classes and thus 5 irreps. From the above conditions one can derive that the degrees of the irreps of $\mathcal{O}$ must be $1, 1, 2, 3, 3$.

Characters

Definition

- For an $n \times n$ matrix $\mathbf{A}$, the sum of the diagonal entries is called its trace, denoted by $\mathrm{tr}(\mathbf{A})$:

$$\mathrm{tr}(\mathbf{A}) = \mathbf{A}_{11} + \mathbf{A}_{22} + \ldots + \mathbf{A}_{nn}.$$

- For a representation **D** of $\mathcal{G}$ the mapping $\chi_{\mathbf{D}} : \mathcal{G} \to \mathbb{C}$ given by

$$\chi_{\mathbf{D}}(g) = \mathrm{tr}(\mathbf{D}(g))$$

is called the character of **D**.

Character value of the inverse element

Let χ be the character of a representation **D** of $\mathcal{G}$.
Then $\chi(g^{-1})$ is the complex conjugate $\chi(g)^*$ of $\chi(g)$.

Invariance of determinants and traces

- The determinant $\det(\mathbf{A})$ is the product of the eigenvalues of $\mathbf{A}$ and the trace $\mathrm{tr}(\mathbf{A})$ is the sum of the eigenvalues, therefore $\det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A})$ do not change under basis transformations.
- For an invertible matrix $\mathbf{X}$ one has $\det(\mathbf{X}^{-1}\mathbf{A}\mathbf{X}) = \det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{X}^{-1}\mathbf{A}\mathbf{X}) = \mathrm{tr}(\mathbf{A})$.

Theorem

Characters are constant on the conjugacy classes of $\mathcal{G}$:
$g' = h^{-1}gh$ implies

$$\begin{aligned}\chi(g') &= \mathrm{tr}(\mathbf{D}(g')) = \mathrm{tr}(\mathbf{D}(h^{-1}gh)) \\ &= \mathrm{tr}(\mathbf{D}(h^{-1})\,\mathbf{D}(g)\,\mathbf{D}(h)) = \mathrm{tr}(\mathbf{D}(g)) = \chi(g).\end{aligned}$$

Criterion for equivalence

Theorem

- Two representations **D** and **D**′ are equivalent if and only if their characters are equal.
- It is sufficient to compare two characters on representatives of the conjugacy classes.
- It is in general not sufficient to compare the character values on generators of the group.

Example

Denote for the three representations

$$\mathbf{D}(g) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{D}'(g) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathbf{D}''(g) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $\mathcal{C}_4$ the corresponding characters by χ, χ', χ''.
Then $\chi(g^{\pm 1}) = \chi'(g^{\pm 1}) = \chi''(g^{\pm 1}) = 0$, but $\chi'(g^2) = \mathrm{tr}(\mathbf{I}_2) = 2$, whereas $\chi(g^2) = \chi''(g^2) = \mathrm{tr}(-\mathbf{I}_2) = -2$.
Hence **D** and **D**′′ are equivalent, but **D**′ is different.

Examples

- For the representation **D** of $\mathcal{G} = 3m$ with

$$\mathbf{D}(g) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \mathbf{D}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  the character $\chi = \chi_{\mathbf{D}}$ has the values

$$\chi(e) = 2, \quad \chi(g) = \chi(g^2) = -1, \quad \chi(h) = \chi(gh) = \chi(g^2 h) = 0$$

  since g^2 is conjugate to g, and gh, g^2h are conjugate to h.
- Let $\mathcal{G}$ be a 3D point group and let **D** be its vector representation with corresponding character $\chi = \chi_{\mathbf{D}}$.
  For a 2-fold rotation $2_{xyz} \in \mathcal{G}$ one has $\chi(2_{xyz}) = -1$, since
  $\mathbf{D}(2_{xyz})$ is equivalent to $\mathbf{D}(2_{001}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
  Analogously, for a reflection $m_{xyz} \in \mathcal{G}$ one has $\chi(m_{xyz}) = 1$,
  since $\mathbf{D}(m_{xyz})$ is equivalent to $\mathbf{D}(m_{001}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

Character table

Definition

Let $\mathcal{G}$ be a finite group with r conjugacy classes, represented by the elements $g_1 = e, g_2, \dots, g_r$ and let $\mathbf{D}^{(1)}, \dots, \mathbf{D}^{(r)}$ be the different irreps of $\mathcal{G}$.
Then the character table of $\mathcal{G}$ is the $r \times r$ matrix $\mathbf{X} = \mathbf{X}(\mathcal{G})$ with

$$\mathbf{X}_{ij} = \chi_{\mathbf{D}^{(i)}}(g_j).$$

The rows of the character table are usually labelled by names for the irreps (which may be just numbers) and the columns are labelled by representatives for the conjugacy classes.

The character table may be augmented with additional information, e.g.:

- for each column the order of the elements;
- for each column the class length of the conjugacy class.

Character table of the 2D point group $3m$

class length	1	2	3	
element order	1	3	2	
	1	3^+	m_{10}	
$\chi^{(1)}$	1	1	1	
$\chi^{(2)}$	1	1	-1	$\leftarrow$ determinant
$\chi^{(3)}$	2	-1	0	$\leftarrow$ vector representation

Character table of the point group 432

class length	1	3	6	8	6	
element order	1	2	2	3	4	
	1	2_{001}	2_{110}	3^+_{111}	4^+_{001}	
$\chi^{(1)}$	1	1	1	1	1	
$\chi^{(2)}$	1	1	-1	1	-1	
$\chi^{(3)}$	2	2	0	-1	0	
$\chi^{(4)}$	3	-1	-1	0	1	$\leftarrow$ vector representation
$\chi^{(5)}$	3	-1	1	0	-1	

Scalar product of characters

Definition

For two characters χ, ψ of $\mathcal{G}$, the scalar product of χ and ψ is defined as

$$(\chi, \psi)_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi(g)\psi(g^{-1}) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi(g)\psi(g)^*$$

Alternative formula

If $g_1, \ldots, g_r$ are representatives for the conjugacy classes of $\mathcal{G}$ and if $|C_j|$ is the number of elements in the conjugacy class of g_j, then the scalar product can be written as

$$(\chi, \psi)_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{j=1}^{r} |C_j| \, \chi(g_j)\psi(g_j)^*.$$

For this version it is useful to augment the character table with the class lengths.

Example: $\mathcal{G} = 2mm$

Character table of the 2D point group $2mm$ with the character ψ of the vector representation appended.

class length	1	1	1	1
element order	1	2	2	2
	1	2	m_{10}	m_{01}
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	-1	-1	1
$\chi^{(3)}$	1	1	-1	-1
$\chi^{(4)}$	1	-1	1	-1
ψ	2	-2	0	0

$$(\chi^{(2)}, \chi^{(3)})_{\mathcal{G}} = \tfrac{1}{4}(1 \cdot 1 + (-1) \cdot 1 + (-1) \cdot (-1) + 1 \cdot (-1)) = \tfrac{1}{4} \cdot 0 = 0$$

$$(\chi^{(2)}, \psi)_{\mathcal{G}} = \tfrac{1}{4}(1 \cdot 2 + (-1) \cdot (-2) + (-1) \cdot 0 + 1 \cdot 0) = \tfrac{1}{4} \cdot 4 = 1$$

$$(\psi, \psi)_{\mathcal{G}} = \tfrac{1}{4}(2 \cdot 2 + (-2) \cdot (-2) + 0 \cdot 0 + 0 \cdot 0) = \tfrac{1}{4} \cdot 8 = 2$$

Example: $\mathcal{G} = 432$

class length	1	3	6	8	6
element order	1	2	2	3	4
	1	2_{001}	2_{110}	3^{+}_{111}	4^{+}_{001}
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	1	-1
$\chi^{(3)}$	2	2	0	-1	0
$\chi^{(4)}$	3	-1	-1	0	1
$\chi^{(5)}$	3	-1	1	0	-1

$$(\chi^{(3)},\chi^{(4)})_{\mathcal{G}} = \frac{1}{24}(1\cdot 2\cdot 3+3\cdot 2\cdot(-1)+6\cdot 0\cdot(-1)+8\cdot(-1)\cdot 0+6\cdot 0\cdot 1) = \frac{1}{24}\cdot 0 = 0$$

$$(\chi^{(4)},\chi^{(4)})_{\mathcal{G}} = \frac{1}{24}(1\cdot 3\cdot 3+3\cdot(-1)\cdot(-1)+6\cdot(-1)\cdot(-1)+8\cdot 0\cdot 0+6\cdot 1\cdot 1) = \frac{1}{24}\cdot 24 = 1$$

Exercise: $\mathcal{G} = 3m$ of order 6

class length	1	2	3
element order	1	3	2
	1	3^+	m_{10}
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	1	-1
$\chi^{(3)}$	2	-1	0
ψ	3	0	1

Compute the scalar products $(\chi^{(i)}, \chi^{(3)})_{\mathcal{G}}$ and $(\chi^{(i)}, \psi)_{\mathcal{G}}$ for $i = 1, 2, 3$.
(The character ψ belongs to the representation we decomposed earlier.)

Answer

$$(\chi^{(1)}, \chi^{(3)})_{\mathcal{G}} = \tfrac{1}{6}(1 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot (-1) + 3 \cdot 1 \cdot 0) = \tfrac{1}{6} \cdot 0 = 0$$
$$(\chi^{(2)}, \chi^{(3)})_{\mathcal{G}} = \tfrac{1}{6}(1 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot (-1) + 3 \cdot (-1) \cdot 0) = \tfrac{1}{6} \cdot 0 = 0$$
$$(\chi^{(3)}, \chi^{(3)})_{\mathcal{G}} = \tfrac{1}{6}(1 \cdot 2 \cdot 2 + 2 \cdot (-1) \cdot (-1) + 3 \cdot 0 \cdot 0) = \tfrac{1}{6} \cdot 6 = 1$$

$$(\chi^{(1)}, \psi)_{\mathcal{G}} = \tfrac{1}{6}(1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3 \cdot 1 \cdot 1) = \tfrac{1}{6} \cdot 6 = 1$$
$$(\chi^{(2)}, \psi)_{\mathcal{G}} = \tfrac{1}{6}(1 \cdot 1 \cdot 3 + 2 \cdot 1 \cdot 0 + 3 \cdot (-1) \cdot 1) = \tfrac{1}{6} \cdot 0 = 0$$
$$(\chi^{(3)}, \psi)_{\mathcal{G}} = \tfrac{1}{6}(1 \cdot 2 \cdot 3 + 2 \cdot (-1) \cdot 0 + 3 \cdot 0 \cdot 1) = \tfrac{1}{6} \cdot 6 = 1$$

Orthogonality relations

Row orthogonality

Let $\mathcal{G}$ be a finite group with irreducible characters $\chi^{(1)}, \ldots, \chi^{(r)}$. Then the irreducible characters of $\mathcal{G}$ form an orthonormal basis w.r.t. $(\cdot, \cdot)_{\mathcal{G}}$, i.e.:

$$(\chi^{(i)}, \chi^{(j)})_{\mathcal{G}} = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \chi^{(i)}(g) \chi^{(j)}(g)^* = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Column orthogonality

The columns of the character table of $\mathcal{G}$ form an orthogonal system:

$$\sum_{i=1}^{r} \chi^{(i)}(g_j) \chi^{(i)}(g_k)^* = \begin{cases} |\mathcal{G}|/|C_j| & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

where $|C_j|$ is the class length of the conjugacy class with representative g_j.

The magic formula

Irreducibility criterion

Let χ be the character of a representation $\mathbf{D}$ of $\mathcal{G}$.
Then $\mathbf{D}$ is irreducible if and only if its character has norm 1, i.e. if

$$\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi^{(i)}(g)\chi^{(i)}(g)^* = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}|\chi^{(i)}(g)|^2 = 1.$$

Magic formula.

Let $\mathcal{G}$ be a finite group with irreps $\mathbf{D}^{(1)},\dots,\mathbf{D}^{(r)}$ and corresponding irreducible characters $\chi^{(1)},\dots,\chi^{(r)}$ and let $\mathbf{D}$ be an arbitrary representation of $\mathcal{G}$ with character $\chi = \chi_{\mathbf{D}}$.
Then the multiplicity m_i with which $\mathbf{D}^{(i)}$ occurs in the decomposition $\mathbf{D} = m_1\mathbf{D}^{(1)} \oplus \dots \oplus m_r\mathbf{D}^{(r)}$ into irreps and so that $\chi = \sum_{i=1}^r m_i\chi^{(i)}$ is the scalar product

$$m_i = (\chi, \chi^{(i)})_{\mathcal{G}} = \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}}\chi(g)\,\chi^{(i)}(g)^*.$$

Example.

We want to decompose the character χ of $\mathcal{O}$ which is appended to the character table:

$\lvert C_j \rvert$	1	3	6	8	6
	1	2_{001}	2_{110}	3^+_{111}	4^+_{001}
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	−1	1	−1
$\chi^{(3)}$	2	2	0	−1	0
$\chi^{(4)}$	3	−1	−1	0	1
$\chi^{(5)}$	3	−1	1	0	−1
χ	12	4	0	0	0

Instead of multiplying the columns of the character table by the $|C_j|$, we multiply the components of χ by $|C_j|$:

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix} = \frac{1}{24} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ 2 & 2 & 0 & -1 & 0 \\ 3 & -1 & -1 & 0 & 1 \\ 3 & -1 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 12 \cdot 1 \\ 4 \cdot 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

Exercise

Three characters $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$ of the symmetry group $4mm$ of the square are appended to its character table:

$\lvert C_j \rvert$	1	1	2	2	2
	1	2	4^+	m_{10}	m_{11}
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	-1	1
$\chi^{(4)}$	1	1	-1	1	-1
$\chi^{(5)}$	2	-2	0	0	0
$\psi^{(1)}$	6	2	0	0	0
$\psi^{(2)}$	10	6	-2	-2	0
$\psi^{(3)}$	11	7	-3	-3	-3

Determine the multiplicities with which the irreps $\chi^{(1)}, \dots, \chi^{(5)}$ occur in $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$.

Answer

$$\psi^{(1)}:\begin{pmatrix}m_1\\m_2\\m_3\\m_4\\m_5\end{pmatrix}=\frac{1}{8}\begin{pmatrix}1&1&1&1&1\\1&1&1&-1&-1\\1&1&-1&-1&1\\1&1&-1&1&-1\\2&-2&0&0&0\end{pmatrix}\begin{pmatrix}6\\2\\0\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$$

$$\psi^{(2)}:\begin{pmatrix}m_1\\m_2\\m_3\\m_4\\m_5\end{pmatrix}=\frac{1}{8}\begin{pmatrix}1&1&1&1&1\\1&1&1&-1&-1\\1&1&-1&-1&1\\1&1&-1&1&-1\\2&-2&0&0&0\end{pmatrix}\begin{pmatrix}10\\6\\-4\\-4\\0\end{pmatrix}=\begin{pmatrix}1\\2\\3\\2\\1\end{pmatrix}$$

$$\psi^{(3)}:\begin{pmatrix}m_1\\m_2\\m_3\\m_4\\m_5\end{pmatrix}=\frac{1}{8}\begin{pmatrix}1&1&1&1&1\\1&1&1&-1&-1\\1&1&-1&-1&1\\1&1&-1&1&-1\\2&-2&0&0&0\end{pmatrix}\begin{pmatrix}11\\7\\-6\\-6\\-6\end{pmatrix}=\begin{pmatrix}0\\3\\3\\3\\1\end{pmatrix}$$