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and

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Overview

Representations of point groups

- direct products of irreps
- symmetrizations
- irreps of direct products of groups
- irreps of cyclic groups
- irreps of dihedral groups
- character tables and irreps of point groups

Kronecker product

Question

The direct sum representation $\mathbf{D}^{(1)} \oplus \mathbf{D}^{(2)}$ has as its character the sum $\chi_{\mathbf{D}^{(1)}} + \chi_{\mathbf{D}^{(2)}}$ of the corresponding characters.
Is there also a representation that has as its character the product $\chi_{\mathbf{D}^{(1)}} \cdot \chi_{\mathbf{D}^{(2)}}$ of the characters?

Definition

For two matrices $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{m\times m}$ the Kronecker product or direct product $\mathbf{A} \otimes \mathbf{B}$ of $\mathbf{A}$ and $\mathbf{B}$ is the $nm \times nm$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} \mathbf{A}_{11}\mathbf{B} & \mathbf{A}_{12}\mathbf{B} & \cdots & \mathbf{A}_{1n}\mathbf{B} \\ \mathbf{A}_{21}\mathbf{B} & \mathbf{A}_{22}\mathbf{B} & \cdots & \mathbf{A}_{2n}\mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{n1}\mathbf{B} & \mathbf{A}_{n2}\mathbf{B} & \cdots & \mathbf{A}_{nn}\mathbf{B} \end{pmatrix}.$$

where $\mathbf{A}_{ij}\mathbf{B}$ is the $m \times m$ matrix obtained by multiplying all elements of $\mathbf{B}$ by $\mathbf{A}_{ij}$ (thus $\mathbf{A}_{ij} \cdot \mathbf{B}_{kl} = (\mathbf{A} \otimes \mathbf{B})_{(i-1)m+k,(j-1)m+l}$).

Example

The Kronecker products $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ of the matrices

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \text{ are}$$

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} 0\,\mathbf{B} & (-1)\,\mathbf{B} \\ 1\,\mathbf{B} & (-1)\,\mathbf{B} \end{pmatrix} = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{array}\right) \text{ and}$$

$$\mathbf{B} \otimes \mathbf{A} = \begin{pmatrix} 0\,\mathbf{A} & 0\,\mathbf{A} & (-1)\mathbf{A} \\ 1\,\mathbf{A} & 0\,\mathbf{A} & 0\mathbf{A} \\ 0\,\mathbf{A} & (-1)\,\mathbf{A} & 0\mathbf{A} \end{pmatrix} = \left(\begin{array}{cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ \hline 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \end{array}\right).$$

Exercise

Determine the Kronecker products $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{B} \otimes \mathbf{A}$ of the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Answer

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} 1\,\mathbf{B} & (-1)\,\mathbf{B} \\ 1\,\mathbf{B} & 0\,\mathbf{B} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

$$\mathbf{B} \otimes \mathbf{A} = \begin{pmatrix} 1\,\mathbf{A} & 1\,\mathbf{A} \\ 1\,\mathbf{A} & (-1)\,\mathbf{A} \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & -1 & 1 & -1 \\ 1 & 0 & 1 & 0 \\ \hline 1 & -1 & -1 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

Definition

For two representations $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ of a group $\mathcal{G}$, defining

$$\mathbf{D}(g) = \mathbf{D}^{(1)}(g) \otimes \mathbf{D}^{(2)}(g)$$

gives a representation of $\mathcal{G}$ called the direct product or tensor product of $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$ and denoted by $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(2)}$.
The character $\chi_{\mathbf{D}}$ is the product of the characters of $\mathbf{D}^{(1)}$ and $\mathbf{D}^{(2)}$, i.e.

$$\chi_{\mathbf{D}}(g) = \chi_{\mathbf{D}^{(1)} \otimes \mathbf{D}^{(2)}}(g) = \chi_{\mathbf{D}^{(1)}}(g) \cdot \chi_{\mathbf{D}^{(2)}}(g).$$

Two crucial properties of the Kronecker product

- $(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = \mathbf{AC} \otimes \mathbf{BD}$ ensures that $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(2)}$ is a homomorphism and thus a representation;
- $\mathrm{tr}(\mathbf{A} \otimes \mathbf{B}) = \mathrm{tr}(\mathbf{A}) \cdot \mathrm{tr}(\mathbf{B})$ yields the formula for the character of $\mathbf{D}^{(1)} \otimes \mathbf{D}^{(2)}$.

Interpretation of the direct product

- Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces with bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$, respectively.
- Denote a pair $(\mathbf{v}_i, \mathbf{w}_j)$ of basis vectors by $\mathbf{v}_i \otimes \mathbf{w}_j$ and take these pairs as basis

$$\mathbf{v}_1 \otimes \mathbf{w}_1, \mathbf{v}_1 \otimes \mathbf{w}_2, \ldots, \mathbf{v}_1 \otimes \mathbf{w}_m, \mathbf{v}_2 \otimes \mathbf{w}_1, \mathbf{v}_2 \otimes \mathbf{w}_2, \ldots, \mathbf{v}_2 \otimes \mathbf{w}_m, \\ \ldots \mathbf{v}_n \otimes \mathbf{w}_1, \mathbf{v}_n \otimes \mathbf{w}_2, \ldots, \mathbf{v}_n \otimes \mathbf{w}_m$$

  of a new vector space $\mathbb{V} \otimes \mathbb{W}$, called the tensor product or direct product of $\mathbb{V}$ and $\mathbb{W}$.
- If $\mathcal{G}$ acts on $\mathbb{V}$ and $\mathbb{W}$ via representations $\mathsf{D}^{(1)}$ and $\mathsf{D}^{(2)}$ with respect to the bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$, respectively, then letting $\mathcal{G}$ act on $\mathbb{V} \otimes \mathbb{W}$ by

$$g(\mathbf{v} \otimes \mathbf{w}) = \mathsf{D}^{(1)}(g)\mathbf{v} \otimes \mathsf{D}^{(2)}(g)\mathbf{w}$$

  gives precisely the direct product representation $\mathsf{D} = \mathsf{D}^{(1)} \otimes \mathsf{D}^{(2)}$ of $\mathcal{G}$ on $\mathbb{V} \otimes \mathbb{W}$ with respect to the basis given above.

Symmetrizations of the tensor square

Let $\mathbb{V}$ be a vector space with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

- The symmetrized square $[\mathbb{V}]^2$ is the subspace of the tensor square $\mathbb{V} \otimes \mathbb{V}$ spanned by the symmetric pairs:

  $$[\mathbb{V}]^2 = \langle \mathbf{v} \otimes \mathbf{v}' \in \mathbb{V} \otimes \mathbb{V} \mid \mathbf{v} \otimes \mathbf{v}' = \mathbf{v}' \otimes \mathbf{v} \rangle,$$

  it has the basis

  $$\{\mathbf{v}_i \otimes \mathbf{v}_j + \mathbf{v}_j \otimes \mathbf{v}_i, 1 \leq i \leq j \leq n\}$$

  and its dimension is $\frac{1}{2}n(n+1)$.
- The antisymmetrized square $\{\mathbb{V}\}^2$ is the subspace of $\mathbb{V} \otimes \mathbb{V}$ spanned by the antisymmetric pairs:

  $$\{\mathbb{V}\}^2 = \langle \mathbf{v} \otimes \mathbf{v}' \in \mathbb{V} \otimes \mathbb{V} \mid \mathbf{v} \otimes \mathbf{v}' = -\mathbf{v}' \otimes \mathbf{v} \rangle,$$

  it has the basis

  $$\{\mathbf{v}_i \otimes \mathbf{v}_j - \mathbf{v}_j \otimes \mathbf{v}_i, 1 \leq i < j \leq n\}$$

  and its dimension is $\frac{1}{2}n(n-1)$.

Symmetrizations of a representation

If **D** is a representation of $\mathcal{G}$ on $\mathbb{V}$, then $\mathbf{D} \otimes \mathbf{D}$ is its representation on the tensor square $\mathbb{V} \otimes \mathbb{V}$.

- The subspaces $[\mathbb{V}]^2$ and $\{\mathbb{V}\}^2$ of $\mathbb{V} \otimes \mathbb{V}$ are invariant under $\mathcal{G}$.
- The representation of $\mathcal{G}$ on $[\mathbb{V}]^2$ is called the symmetrized square and is denoted by $[\mathbf{D}]^2$.
- The representation of $\mathcal{G}$ on $\{\mathbb{V}\}^2$ is called the antisymmetrized square and is denoted by $\{\mathbf{D}\}^2$.
- The tensor square decomposes as $\mathbf{D} \otimes \mathbf{D} = [\mathbf{D}]^2 \oplus \{\mathbf{D}\}^2$.
- The symmetrizations $[\mathbf{D}]^2$ and $\{\mathbf{D}\}^2$ can be reducible or irreducible.

Characters of symmetrizations

Let **D** be a representation of $\mathcal{G}$ with character χ.

- The character $[\chi]^2$ of the symmetrized square $[\mathbf{D}]^2$ is

$$[\chi]^2(g) = \frac{1}{2}\left(\chi^2(g) + \chi(g^2)\right).$$

- The character $\{\chi\}^2$ of the antisymmetrized square $\{\mathbf{D}\}^2$ is

$$\{\chi\}^2(g) = \frac{1}{2}\left(\chi^2(g) - \chi(g^2)\right).$$

Example: $4mm$

	1	2	4^+	m_{10}	m_{11}
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	1	1	-1	-1	1
$\chi^{(4)}$	1	1	-1	1	-1
$\chi^{(5)}$	2	-2	0	0	0

$[\chi^{(5)}]^2$	3	3	-1	1	1
$\{\chi^{(5)}\}^2$	1	1	1	-1	-1

The decomposition of the symmetrized square into irreps is $[\chi^{(5)}]^2 = \chi^{(1)} + \chi^{(3)} + \chi^{(4)}$, the antisymmetrized square is equal to the irrep $\chi^{(2)}$.

Exercise: $3m$

	1	3^+	m_{10}
$\chi^{(1)}$	1	1	1
$\chi^{(2)}$	1	1	-1
$\chi^{(3)}$	2	-1	0

Determine the symmetrized square $[\chi^{(3)}]^2$ and the antisymmetrized square $\{\chi^{(3)}\}^2$ of $\chi^{(3)}$ and decompose them into irreps.

Answer

$[\chi^{(3)}]^2$	3	0	1	$= \chi^{(1)} + \chi^{(3)}$
$\{\chi^{(3)}\}^2$	1	1	-1	$= \chi^{(2)}$

Irreps of direct products of groups.

Definition

For two groups $\mathcal{G}_1$ and $\mathcal{G}_2$, the direct product $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ is the set of pairs (g_1, g_2) with $g_1 \in \mathcal{G}_1$ and $g_2 \in \mathcal{G}_2$ and componentwise operation:

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2').$$

In many cases, a group $\mathcal{G}$ can be written as a direct product $\mathcal{G} = \mathcal{H}_1 \times \mathcal{H}_2$ of two of its subgroups. In this case one simply writes $g = h_1 \cdot h_2$ instead of $g = (h_1, h_2)$.

Theorem

Let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ be the direct product of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$. Let $\mathbf{D}_1^{(i)}$ for $1 \leq i \leq r$ be the irreps of $\mathcal{G}_1$ and let $\mathbf{D}_2^{(j)}$ for $1 \leq j \leq s$ be the irreps of $\mathcal{G}_2$.

Then the irreps of $\mathcal{G}$ are given by the direct product representations $\mathbf{D}^{(ij)}$ with

$$\mathbf{D}^{(ij)}((g_1, g_2)) = \mathbf{D}_1^{(i)}(g_1) \otimes \mathbf{D}_2^{(j)}(g_2)$$

for $1 \leq i \leq r$ and $1 \leq j \leq s$.

Example

- The direct product of the point groups $4 = \langle 4^+_{001} \rangle$ and $\bar{1}$ is the point group $4/m$.
- The tensor product $\mathbf{D} = \mathbf{D}^{(1)} \otimes \mathbf{D}^{(2)}$ of the representations $\mathbf{D}^{(1)}$ of 4 and $\mathbf{D}^{(2)}$ of $\bar{1}$ given by

$$\mathbf{D}^{(1)}(4^+_{001}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D}^{(2)}(\bar{1}) = \begin{pmatrix} -1 \end{pmatrix}$$

is a representation of $4/m$ with $\mathbf{D}(4^+_{001}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$\mathbf{D}(\bar{1}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\mathbf{D}(4^+_{001} \cdot \bar{1}) = \mathbf{D}(\bar{4}^+_{001}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Example (ctd.)

- The direct product of the point groups $4 = \langle 4^+_{001} \rangle$ and $\bar{1}$ is the point group $4/m$.
- The tensor product $\mathbf{D} = \mathbf{D}^{(1)} \otimes \mathbf{D}^{(2)}$ of the representations $\mathbf{D}^{(1)}$ of 4 and $\mathbf{D}^{(2)}$ of $\bar{1}$ given by

$$\mathbf{D}^{(1)}(4^+_{001}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{D}^{(2)}(\bar{1}) = \begin{pmatrix} -1 \end{pmatrix}$$

is a representation of $4/m$ with $\mathbf{D}(4^+_{001}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\mathbf{D}(\bar{1}) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{D}(\bar{4}^+_{001}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{D}(4^-_{001})$.

The representation $\mathbf{D}$ has $2_{001} \cdot \bar{1} = m_{001}$ in its kernel, since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Character tables of direct products of groups

Theorem

Let $\mathbf{X}_1$ be the character table of $\mathcal{G}_1$ for conjugacy class representatives $g_1, \ldots, g_r$ and irreps $\mathbf{D}_1^{(1)}, \ldots, \mathbf{D}_1^{(r)}$ of $\mathcal{G}_1$ and let $\mathbf{X}_2$ be the character table of $\mathcal{G}_2$ for conjugacy class representatives $g_1', \ldots, g_s'$ and irreps $\mathbf{D}_2^{(1)}, \ldots, \mathbf{D}_2^{(s)}$ of $\mathcal{G}_2$.

Taking the conjugacy class representatives of $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ in the order

$$\begin{aligned} (g_1, g_1'), (g_1, g_2'), \ldots, (g_1, g_s'), \quad & (g_2, g_1'), (g_2, g_2'), \ldots, (g_2, g_s'), \\ \ldots \quad & (g_r, g_1'), (g_r, g_2'), \ldots, (g_r, g_s') \end{aligned}$$

and the irreps of $\mathcal{G}$ in the order

$$\begin{aligned} \mathbf{D}^{(11)}, \mathbf{D}^{(12)}, \ldots, \mathbf{D}^{(1s)}, \quad & \mathbf{D}^{(21)}, \mathbf{D}^{(22)}, \ldots, \mathbf{D}^{(2s)}, \\ \ldots \quad & \mathbf{D}^{(r1)}, \mathbf{D}^{(r2)}, \ldots, \mathbf{D}^{(rs)} \end{aligned}$$

the character table of $\mathcal{G}$ is the Kronecker product $\mathbf{X}_1 \otimes \mathbf{X}_2$.

Example

The 2D point groups $\bar{1}$ and $3m$ have character tables

	1	$\bar{1}$
$\chi_{\mathbf{D}^{(1)}}$	1	1
$\chi_{\mathbf{D}^{(2)}}$	1	-1

	1	m_{10}	3^+
$\chi_{\mathbf{D}^{(1)}}$	1	1	1
$\chi_{\mathbf{D}^{(2)}}$	1	-1	1
$\chi_{\mathbf{D}^{(3)}}$	2	0	-1

The direct product $\bar{1} \times 3m = 6mm$ therefore has the character table

	1	m_{10}	3^+	$\bar{1}$	m_{12}	6^+
$\chi_{\mathbf{D}^{(11)}}$	1	1	1	1	1	1
$\chi_{\mathbf{D}^{(12)}}$	1	-1	1	1	-1	1
$\chi_{\mathbf{D}^{(13)}}$	2	0	-1	2	0	-1
$\chi_{\mathbf{D}^{(21)}}$	1	1	1	-1	-1	-1
$\chi_{\mathbf{D}^{(22)}}$	1	-1	1	-1	1	-1
$\chi_{\mathbf{D}^{(23)}}$	2	0	-1	-2	0	1

Direct products with $\mathcal{C}_2$

If $\mathbf{X}$ is the character table of $\mathcal{G}$, then the character table of $\mathcal{C}_2 \times \mathcal{G}$ and $\mathcal{G} \times \mathcal{C}_2$ (which are isomorphic to each other) is

$$\begin{pmatrix} \mathbf{X} & \mathbf{X} \\ \mathbf{X} & -\mathbf{X} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \mathbf{X}$$

since $\mathcal{C}_2 = \{1, g\}$ has character table

	1	g
$\chi_{\mathbf{D}^{(1)}}$	1	1
$\chi_{\mathbf{D}^{(2)}}$	1	-1

Overview of point groups

abstract group	order	point groups
$\mathcal{C}_1$	1	1
$\mathcal{C}_2$	2	2, m, $\bar{1}$
$\mathcal{C}_3$	3	3
$\mathcal{C}_4$	4	4, $\bar{4}$
$\mathcal{C}_6 \cong \mathcal{C}_3 \times \mathcal{C}_2$	6	$\bar{3}$, 6, $\bar{6}$
$\mathcal{C}_{4h} \cong \mathcal{C}_4 \times \mathcal{C}_2$	8	$4/m$
$\mathcal{C}_{6h} \cong \mathcal{C}_6 \times \mathcal{C}_2$	12	$6/m$
$\mathcal{D}_2 \cong \mathcal{C}_2 \times \mathcal{C}_2$	4	$2/m$, 222, $mm2$
$\mathcal{D}_3$	6	32, $3m$
$\mathcal{D}_4$	8	422, $4mm$, $\bar{4}2m$
$\mathcal{D}_6 \cong \mathcal{D}_{3h} \cong \mathcal{D}_3 \times \mathcal{C}_2$	12	$\bar{3}m$, 622, $6mm$, $\bar{6}2m$
$\mathcal{D}_{2h} \cong \mathcal{D}_2 \times \mathcal{C}_2 \cong \mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$	8	mmm
$\mathcal{D}_{4h} \cong \mathcal{D}_4 \times \mathcal{C}_2$	16	$4/mmm$
$\mathcal{D}_{6h} \cong \mathcal{D}_6 \times \mathcal{C}_2$	24	$6/mmm$
$\mathcal{T}$	12	23
$\mathcal{T}_h \cong \mathcal{T} \times \mathcal{C}_2$	24	$m\bar{3}$
$\mathcal{O}$	24	432, $\bar{4}3m$
$\mathcal{O}_h \cong \mathcal{O} \times \mathcal{C}_2$	48	$m\bar{3}m$

To do list

- Given the character table of a group $\mathcal{G}$, we know how to construct the character table and irreps of $\mathcal{C}_2 \times \mathcal{G} = \mathcal{G} \times \mathcal{C}_2$.
- In order to derive the character tables and irreps of all the point groups, it is therefore sufficient to know the character tables and irreps of the following (types of) groups:
  - cyclic groups $\mathcal{C}_n$, especially for $n = 2, 3, 4, 6$
  - dihedral groups $\mathcal{D}_n$, especially for $n = 2, 3, 4, 6$, realized by the rotation groups 222, 32, 422, 622
  - the tetrahedral group $\mathcal{T}$, realized by 23
  - the octahedral group $\mathcal{O}$, realized by 432

Irreps of cyclic groups

Reminder

The irreps of abelian groups, and therefore also of cyclic groups are 1-dimensional.

Notation

The complex number $e^{\frac{2\pi i}{n}}$ is often denoted by ζ_n and is called a primitive n-th root of unity, since $(e^{\frac{2\pi i}{n}})^n = e^{2\pi i} = 1$.

Theorem

The irreps of a cyclic group $\mathcal{C}_n$ of order n generated by g are of the form

$$\mathbf{D}^{(k)}(g) = \left(e^{2\pi i \frac{k}{n}}\right) = \left(\zeta_n^k\right) \text{ with } 0 \le k < n,$$

since for $\mathbf{D}^{(k)}(g) = \left(z\right)$ one requires $z^n = 1$.
For an arbitrary element $g^a \in \mathcal{C}_n$ one has

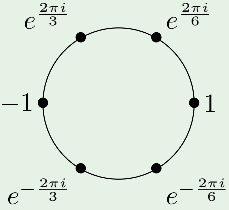
$$\mathbf{D}^{(k)}(g^a) = \left(e^{2\pi i a \frac{k}{n}}\right) = \left(\zeta_n^{ak}\right).$$

Examples: The cyclic groups $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_6$

$\mathcal{C}_3$	e	g	g^2
$\mathbf{D}^{(0)}$	1	1	1
$\mathbf{D}^{(1)}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$
$\mathbf{D}^{(2)}$	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$

$\mathcal{C}_4$	e	g	g^2	g^3
$\mathbf{D}^{(0)}$	1	1	1	1
$\mathbf{D}^{(1)}$	1	i	-1	$-i$
$\mathbf{D}^{(2)}$	1	-1	1	-1
$\mathbf{D}^{(3)}$	1	$-i$	-1	i

$\mathcal{C}_6$	e	g	g^2	g^3	g^4	g^5
$\mathbf{D}^{(0)}$	1	1	1	1	1	1
$\mathbf{D}^{(1)}$	1	$e^{\frac{2\pi i}{6}}$	$e^{\frac{2\pi i}{3}}$	-1	$e^{-\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{6}}$
$\mathbf{D}^{(2)}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$
$\mathbf{D}^{(3)}$	1	-1	1	-1	1	-1
$\mathbf{D}^{(4)}$	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
$\mathbf{D}^{(5)}$	1	$e^{-\frac{2\pi i}{6}}$	$e^{-\frac{2\pi i}{3}}$	-1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{6}}$



Direct products of cyclic groups

Notation

From now on, we will simply write $g_1 g_2$ for the elements (g_1, g_2) in a direct product $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$.

Direct products of two cyclic groups

Let $\mathcal{C}_n$ be generated by g_1 and $\mathcal{C}_m$ generated by g_2, then the irreps $\mathbf{D}^{(kl)}$ $(0 \leq k < n$, $0 \leq l < m)$ of $\mathcal{C}_n \times \mathcal{C}_m$ are determined by the images of g_1 and g_2:

$$\mathbf{D}^{(kl)}(g_1) = \left(e^{2\pi i \frac{k}{n}}\right), \quad \mathbf{D}^{(kl)}(g_2) = \left(e^{2\pi i \frac{l}{m}}\right).$$

The representation of a general element $g_1^a g_2^b$ is

$$\mathbf{D}^{(kl)}(g_1^a g_2^b) = \left(e^{2\pi i (a\frac{k}{n} + b\frac{l}{m})}\right).$$

Example: $\mathcal{C}_2 \times \mathcal{C}_4$

Let $\mathcal{C}_2 = \{+1, -1\}$ and $\mathcal{C}_4 = \{e, g, g^2, g^3\}$.
Then the irreps of $\mathcal{C}_2 \times \mathcal{C}_4$ are determined by

$$\mathbf{D}^{(0k)}(-1) = \left(1\right), \ \mathbf{D}^{(1k)}(-1) = \left(-1\right), \quad \mathbf{D}^{(0k)}(g) = \mathbf{D}^{(1k)}(g) = \left(i^k\right)$$

and the character table is as follows:

	e	g	g^2	g^3	$-e$	$-g$	$-g^2$	$-g^3$
$\mathbf{D}^{(00)}$	1	1	1	1	1	1	1	1
$\mathbf{D}^{(01)}$	1	i	-1	$-i$	1	i	-1	$-i$
$\mathbf{D}^{(02)}$	1	-1	1	-1	1	-1	1	-1
$\mathbf{D}^{(03)}$	1	$-i$	-1	i	1	$-i$	-1	i
$\mathbf{D}^{(10)}$	1	1	1	1	-1	-1	-1	-1
$\mathbf{D}^{(11)}$	1	i	-1	$-i$	-1	$-i$	1	i
$\mathbf{D}^{(12)}$	1	-1	1	-1	-1	1	-1	1
$\mathbf{D}^{(13)}$	1	$-i$	-1	i	-1	i	1	$-i$

Exercise

Let $\mathcal{C}_2 = \{+1, -1\}$ and $\mathcal{C}_3 = \{e, g, g^2\}$.
Determine the irreps and the character table of $\mathcal{C}_2 \times \mathcal{C}_3$.

Answer

The irreps of $\mathcal{C}_2 \times \mathcal{C}_3$ are determined by

$$\mathbf{D}^{(0k)}(-1) = \left(1\right), \ \mathbf{D}^{(1k)}(-1) = \left(-1\right), \quad \mathbf{D}^{(0k)}(g) = \mathbf{D}^{(1k)}(g) = \left(e^{\frac{2\pi i k}{3}}\right)$$

and the character table is:

	e	g	g^2	$-e$	$-g$	$-g^2$
$\mathbf{D}^{(00)}$	1	1	1	1	1	1
$\mathbf{D}^{(01)}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$
$\mathbf{D}^{(02)}$	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
$\mathbf{D}^{(10)}$	1	1	1	-1	-1	-1
$\mathbf{D}^{(11)}$	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	-1	$-e^{\frac{2\pi i}{3}}$	$-e^{-\frac{2\pi i}{3}}$
$\mathbf{D}^{(12)}$	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	-1	$-e^{-\frac{2\pi i}{3}}$	$-e^{\frac{2\pi i}{3}}$

Note that $\mathcal{C}_2 \times \mathcal{C}_3$ is isomorphic to $\mathcal{C}_6$, since $-g$ has order 6.

Two cyclic groups of the same order

The irreps of $\mathcal{C}_n \times \mathcal{C}_n$ are given by

$$\mathbf{D}^{(kl)}(g_1^a g_2^b) = \left(e^{\frac{2\pi i}{n}(ak+bl)}\right) \qquad \text{with } 0 \leq k, l < n.$$

For $\mathbf{k} = \begin{pmatrix} k \\ l \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} a \\ b \end{pmatrix}$ this can conveniently be written as

$$\mathbf{D}^{(kl)}(g_1^a g_2^b) = \left(e^{\frac{2\pi i}{n}\mathbf{k}\cdot\mathbf{t}}\right).$$

Three cyclic groups of the same order

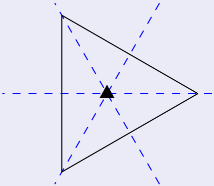
The irreps of $\mathcal{C}_n \times \mathcal{C}_n \times \mathcal{C}_n$ are given by

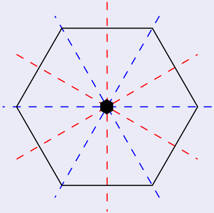
$$\mathbf{D}^{(klm)}(g_1^a g_2^b g_3^c) = \left(e^{\frac{2\pi i}{n}(ak+bl+cm)}\right) \qquad \text{with } 0 \leq k, l, m < n.$$

For $\mathbf{k} = \begin{pmatrix} k \\ l \\ m \end{pmatrix}$ and $\mathbf{t} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ this is $\mathbf{D}^{(kl)}(g_1^a g_2^b g_c^c) = \left(e^{\frac{2\pi i}{n}\mathbf{k}\cdot\mathbf{t}}\right)$.

Dihedral groups.

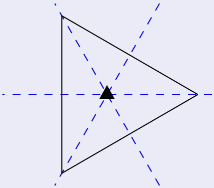
Generators of the dihedral groups

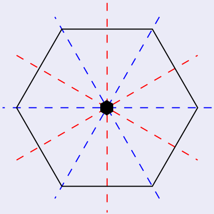




- The dihedral group $\mathcal{D}_n$ of order $2n$ is the symmetry group of a regular n-gon.
- It is generated by a rotation g of order n and a reflection h such that $h^{-1}gh = g^{-1}$.
- The element g generates a cyclic normal subgroup $\mathcal{C}_n \trianglelefteq \mathcal{D}_n$ of index 2 in $\mathcal{D}_n$.

Conjugacy classes of the dihedral groups





- Since h conjugates g^i to $g^{-i} = g^{n-i}$, representatives for the conjugacy classes of rotations are $e, g, \ldots, g^k$ with $k = \frac{n-1}{2}$ for n odd and $k = \frac{n}{2}$ for n even.
- If n is odd, all reflections in $\mathcal{D}_n$ are conjugate, their reflection lines run through a corner and the centre of the opposite side.
- If n is even, there are two conjugacy classes of reflections, represented by h and gh, one with reflection lines running through opposite corners and one through the centers of opposite sides.

Irreps of $\mathcal{D}_n$

- If n is odd, $\mathcal{D}_n$ has 2 irreps of degree 1, they have g in their kernel and send the reflection h either to 1 or to -1.
- If n is even, $\mathcal{D}_n$ has 4 irreps of degree 1, they have g^2 in their kernel and send the reflections h and gh either to 1 or to -1.
- The remaining irreps are 2-dimensional, they are given by

$$\mathbf{E}_k(g) = \begin{pmatrix} e^{\frac{2\pi i}{n}k} & 0 \\ 0 & e^{-\frac{2\pi i}{n}k} \end{pmatrix}, \quad \mathbf{E}_k(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  for $1 \leq k < n/2$.
- For $\theta = \frac{2\pi}{n}$, the corresponding character values are $\chi_{\mathbf{E}_k}(g) = e^{\frac{2\pi i}{n}k} + e^{-\frac{2\pi i}{n}k} = 2\cos(k\theta)$ and $\chi_{\mathbf{E}_k}(h) = 0$.

Example: $\mathcal{D}_6$

- The dihedral group $\mathcal{D}_6$ has conjugacy class representatives e, g, g^2, g^3, h, gh.
- The four 1-dimensional representations are determined by $\chi(h) = \pm 1$ and $\chi(gh) = \pm 1$.
- For the 2-dimensional representations, note that $e^{\frac{2\pi i}{6}} + e^{-\frac{2\pi i}{6}} = 1$ and $e^{\frac{2\pi i}{3}} + e^{-\frac{2\pi i}{3}} = -1$.
- The full character table of $\mathcal{D}_6$ is

	e	g	g^2	g^3	h	gh
$\chi^{(1)}$	1	1	1	1	1	1
$\chi^{(2)}$	1	-1	1	-1	-1	1
$\chi^{(3)}$	1	-1	1	-1	1	-1
$\chi^{(4)}$	1	1	1	1	-1	-1
$\chi_{\mathbf{E}_1}$	2	1	-1	-2	0	0
$\chi_{\mathbf{E}_2}$	2	-1	-1	2	0	0

Exercise

Determine the character table and irreps of the point group $4mm$ (isomorphic to the dihedral group $\mathcal{D}_4$ of order 8) with conjugacy class representatives 1, 2_{001}, 4^+_{001}, m_{100}, m_{110}.

Answer

In the abstract notation we have $4^+_{001} = g$, $2_{001} = g^2$, $m_{100} = h$, $m_{110} = gh$.

	1	2_{001}	4^+_{001}	m_{100}	m_{110}
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	-1	-1	1
$\chi^{(3)}$	1	1	-1	1	-1
$\chi^{(4)}$	1	1	1	-1	-1
$\chi_{\mathbf{E}}$	2	-2	0	0	0

The 2-dimensional representation **E** is

$\mathbf{E}(4^+_{001}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\quad \mathbf{E}(h) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and is equivalent to

$\mathbf{D}(4^+_{001}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\quad \mathbf{D}(h) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Tetrahedral group $\mathcal{T}$

Realization as a 3D rotation group

- The rotation group $\mathcal{T}$ of a tetrahedron is realized by the point group 23, generated by 3^+_{111} and 2_{001}.
- The vector representation

$$\mathbf{D}(3^+_{111}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{D}(2_{001}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  of 23 is irreducible, since the eigenvectors of $\mathbf{D}(3^+_{111})$ are

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{3}} \\ e^{-\frac{2\pi i}{3}} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ e^{-\frac{2\pi i}{3}} \\ e^{\frac{2\pi i}{3}} \end{pmatrix},$$

  but none of these vectors is an eigenvector of $\mathbf{D}(2_{001})$.
- Since the sum of the squares of the degrees of the irreps is 12, the remaining irreps are 3 irreps of degree 1.

Character table of $\mathcal{T}$

- The character table of $\mathcal{T}$ is

	1	2_{001}	3^+_{111}	3^-_{111}
$\chi^{(1)}$	1	1	1	1
$\chi^{(2)}$	1	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$
$\chi^{(3)}$	1	1	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
$\chi_{\mathbf{D}} = \chi^{(4)}$	3	-1	0	0

- The 2-fold rotations in $\mathcal{T}$ form a normal subgroup $\mathcal{H} = \{1, 2_{100}, 2_{010}, 2_{001}\}$ of order 4 and index 3.
- The factor group $\mathcal{T}/\mathcal{H}$ is a cyclic group of order 3, generated by the coset $3^+_{111}\mathcal{H}$.
- The three 1-dimensional irreps of $\mathcal{T}/\mathcal{H}$ are irreps of $\mathcal{T}$ with $\mathcal{H}$ in their kernel.

Octahedral group.

Realization as a 3D rotation group

- The rotation group $\mathcal{O}$ of an octahedron is realized by the point group 432, generated by 2_{110} and 4^+_{001} which contains the tetrahedral group $\mathcal{T}$ as a normal subgroup of index 2.
- The vector representations

$$\mathbf{D}(2_{110}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{D}(4^+_{001}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  of 432 is irreducible, since it extends the vector representation of 23 given before.
- The non-trivial 1-dimensional representation of $\mathcal{O}/\mathcal{T} \cong \mathcal{C}_2$ gives a 1-dimensional irrep of $\mathcal{O}$ which is (1) on the elements of $\mathcal{T}$ and (-1) on the other elements of $\mathcal{O}$.
- The tensor product of this 1-dimensional irrep with the vector representation gives another 3-dimensional irrep of $\mathcal{O}$.

Character table of $\mathcal{O}$

- The character table of $\mathcal{O}$ is

	1	2_{001}	3^{+}_{111}	2_{110}	4^{+}_{001}
$\chi^{(1)}$	1	1	1	1	1
$\chi^{(2)}$	1	1	1	-1	-1
$\chi^{(3)}$	2	2	-1	0	0
$\chi_{\mathbf{D}} = \chi^{(4)}$	3	-1	0	-1	1
$\chi^{(5)}$	3	-1	0	1	-1

- The 2-fold rotations in $\mathcal{T}$ form a normal subgroup $\mathcal{H} = \{1, 2_{100}, 2_{010}, 2_{001}\}$ of order 4 and index 6.
- The factor group $\mathcal{O}/\mathcal{H}$ is isomorphic to a dihedral group of order 6, generated by the cosets $3^{+}_{111}\mathcal{H}$ and $2_{110}\mathcal{H}$.
- The 2-dimensional irrep of $\mathcal{O}/\mathcal{H}$ is an irrep of $\mathcal{O}$ with $\mathcal{H}$ in its kernel.