International Union of Crystallography
Commission on Mathematical and Theoretical Crystallography

Българско Кристалографско Дружество
Bulgarian Crystallographic Society

International school on fundamental crystallography:

Introduction to International Tables for Crystallography,
Vol. A: Space-group symmetry and
Vol. A1: Symmetry relations between space groups

Gulechitza, Bulgaria, 30 September - 5 October 2013
CRYSTALLOGRAPHIC
POINT GROUPS
(short review)

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Universidad del País Vasco, Bilbao, Spain
1. Crystallographic symmetry operations

Symmetry operations of an object

The symmetry operations are isometries, i.e. they are special kind of mappings between an object and its image that leave all distances and angles invariant.

The isometries which map the object onto itself are called symmetry operations of this object. The symmetry of the object is the set of all its symmetry operations.

Crystallographic symmetry operations

If the object is a crystal pattern, representing a real crystal, its symmetry operations are called crystallographic symmetry operations.

The equilateral triangle allows six symmetry operations: rotations by 120° and 240° around its centre, reflections through the three thick lines intersecting the centre, and the identity operation.
GROUP THEORY
(few basic facts)
Crystallographic symmetry operations in the plane

Mirror symmetry operation

Mirror line $m_y$ at $0,y$

Matrix representation

Fixed points

$\begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} x_f \\ y_f \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix}$

$\det \begin{pmatrix} -1 \\ 0 \end{pmatrix} = ?$

$\text{tr} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = ?$
Symmetry operations in the plane
Matrix representations

2-fold rotation

\[
\begin{array}{c|c}
\text{x} & -\text{x} \\
\hline
\text{y} & -\text{y} \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{-x} & \text{x} \\
\hline
\text{-y} & \text{y} \\
\end{array}
\]

\[
\text{det} = ?
\]

\[
\text{tr} = ?
\]

3-fold rotation

\[
\begin{array}{c|c}
\text{x} & -\text{y} \\
\hline
\text{y} & \text{0} \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{0} & -\text{1} \\
\hline
\text{-1} & \text{0} \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
\text{1} & \text{1} \\
\end{array}
\]

\[
\text{det} = ?
\]

\[
\text{tr} = ?
\]
GROUP AXIOMS

1. CLOSURE

\[ g_1 \circ g_2 = g_12 \quad g_1, g_2, g_12 \in G \]

2. IDENTITY

\[ g \circ e = e \circ g = g \]

3. INVERSE ELEMENT

\[ g \circ g^{-1} = e \]

4. ASSOCIATIVITY

\[ (g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) = g_1 \circ g_2 \circ g_3 \]
1. **Order of a group**
   number of elements

2. **Multiplication table**

```
<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>E</td>
<td>A</td>
</tr>
</tbody>
</table>
```

3. **Group generators**
   a set of elements such that each element of the group can be obtained as a product of the generators
4. How to define a group

**Multiplication table**

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>E</td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>A</td>
<td>A</td>
<td>B</td>
<td>E</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>E</td>
<td>A</td>
</tr>
</tbody>
</table>

**Group generators**

A set of elements such that each element of the group can be obtained as a product of the generators
Isomorphic groups

\[ G = \{ g \} \quad \Phi(g) = g' \quad G' = \{ g' \} \]

\[ \Phi^{-1}(g') = g \]

\[ \Phi: G \to G' \quad \Phi^{-1}: G' \to G \]

homomorphic condition

\[ \Phi(g_1) \Phi(g_2) = \Phi(g_1 g_2) \]

-groups with the same multiplication table
Crystallographic Point Groups in 2D

Point group $\mathbf{2} = \{1, 2\}$

- Group axioms?

- Order of $\mathbf{2}$?

- Multiplication table

Motif with symmetry of $\mathbf{2}$

Where is the two-fold point?

- Generators of $\mathbf{2}$?

drawing: M.M. Julian
Foundations of Crystallography
© Taylor & Francis, 2008
Crystallographic Point Groups in 2D

Point group \( m = \{1, m\} \)

Motif with symmetry of \( m \)

- Group axioms?

\[
m \times m = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\]

- Order of \( m \)?

- Multiplication table

\[
\begin{array}{c|cc}
\times & 1 & m_y \\
\hline
1 & 1 & m_y \\
m_y & m_y & 1
\end{array}
\]

- Generators of \( m \)?

Where is the mirror line?

drawing: M.M. Julian
Foundations of Crystallography
© Taylor & Francis, 2008
Isomorphic groups

Point group \(2 = \{1,2\}\)

<table>
<thead>
<tr>
<th>×</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Point group \(m = \{1,m\}\)

<table>
<thead>
<tr>
<th>×</th>
<th>1</th>
<th>(m_y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(m_y)</td>
</tr>
<tr>
<td>(m_y)</td>
<td>(m_y)</td>
<td>1</td>
</tr>
</tbody>
</table>

-groups with the same multiplication table
Crystallographic Point Groups in 2D

Point group \( \mathbf{1} = \{1\} \)

Motif with symmetry of \( \mathbf{1} \)

drawing: M.M. Julian
Foundations of Crystallography
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Problem 2.7

Consider the model of the molecule of the organic semiconductor pentacene (C$_{22}$H$_{14}$):

Determine:

- symmetry operations: matrix and (x,y) presentation
- generators
- multiplication table
Problem 2.8

Consider the symmetry group of the square. Determine:

- symmetry operations: matrix and $(x,y)$ presentation
- generators
- multiplication table
Visualization of Crystallographic Point Groups

- general position diagram
- symmetry elements diagram

Stereographic Projections

Points $P$ in the projection plane
EXAMPLE

Stereographic Projections of $mm2$

Point group $mm2 = \{1, 2_z, m_x, m_y\}$

Molecule of pentacene

Stereographic projections diagrams

general position symmetry elements
Problem 2.8 (cont.)

Stereographic Projections of 4mm

general position diagram

symmetry elements diagram
Consider the symmetry group of the equilateral triangle. Determine:

- symmetry operations: matrix and \((x,y)\) presentation
- general-position and symmetry-elements stereographic projection diagrams;
- generators
- multiplication table
Conjugate elements

Conjugate elements $g_i \sim g_k$ if $\exists g \in G: g^{-1} g_i g = g_k$, where $g, g_i, g_k \in G$

Classes of conjugate elements

$L(g_i) = \{g_j | g^{-1} g_i g = g_j, g \in G\}$

Conjugation-properties

(i) $L(g_i) \cap L(g_j) = \emptyset$, if $g_i \notin L(g_j)$

(ii) $|L(g_i)|$ is a divisor of $|G|$  \hspace{1cm} (iii) $L(e) = \{e\}$

(iv) if $g_i, g_j \in L$, then $(g_i)^k = (g_j)^k = e$
Example (Problem 2.8):

Classes of conjugate elements

The group of the square 4mm

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>4</th>
<th>4⁻¹</th>
<th>$m_x$</th>
<th>$m_+$</th>
<th>$m_y$</th>
<th>$m_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>4⁻¹</td>
<td>$m_x$</td>
<td>$m_+$</td>
<td>$m_y$</td>
<td>$m_-$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>4⁻¹</td>
<td>4</td>
<td>$m_y$</td>
<td>$m_-$</td>
<td>$m_x$</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4⁻¹</td>
<td>2</td>
<td>1</td>
<td>$m_+$</td>
<td>$m_y$</td>
<td>$m_-$</td>
</tr>
<tr>
<td>4⁻¹</td>
<td>4⁻¹</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>$m_-$</td>
<td>$m_x$</td>
<td>$m_+$</td>
</tr>
<tr>
<td>$m_x$</td>
<td>$m_x$</td>
<td>$m_y$</td>
<td>$m_-$</td>
<td>$m_+$</td>
<td>1</td>
<td>4⁻¹</td>
<td>2</td>
</tr>
<tr>
<td>$m_+$</td>
<td>$m_+$</td>
<td>$m_-$</td>
<td>$m_x$</td>
<td>$m_y$</td>
<td>4</td>
<td>1</td>
<td>4⁻¹</td>
</tr>
<tr>
<td>$m_y$</td>
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<td>$m_x$</td>
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<td>$m_-$</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>$m_-$</td>
<td>$m_-$</td>
<td>$m_+$</td>
<td>$m_y$</td>
<td>$m_x$</td>
<td>4⁻¹</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Multiplication table of 4mm

Classes of conjugate elements:

\{1\}, \{2\},\{4,4⁻\},\{m_x,m_y\}, \{m_+,m_-\}
Distribute the symmetry elements of the group \( \text{mm2} = \{1, 2_z, m_x, m_y\} \) in classes of conjugate elements.

\[
\begin{array}{c|cccc}
\times & 1 & 2 & m_x & m_y \\
\hline
1 & 1 & 2 & m_x & m_y \\
2 & 2 & 1 & m_y & m_x \\
m_x & m_x & m_y & 1 & 2 \\
m_y & m_y & m_x & 2 & 1 \\
\end{array}
\]
GROUP-SUBGROUP RELATIONS

I. Subgroups: index, coset decomposition and normal subgroups

II. Conjugate subgroups

III. Group-subgroup graphs
Subgroups: Some basic results (summary)

Subgroup $H < G$

1. $H=\{e,h_1,h_2,\ldots,h_k\} \subset G$
2. $H$ satisfies the group axioms of $G$

Proper subgroups $H < G$, and
trivial subgroup: $\{e\}$, $G$

Index of the subgroup $H$ in $G$: $[i]=|G|/|H|$
(order of $G$)/(order of $H$)

Maximal subgroup $H$ of $G$

NO subgroup $Z$ exists such that:
$H < Z < G$
EXERCISES

Problem 2.11 Consider the group of the square and determine its subgroups

Multiplication table of $4mm$

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>4</th>
<th>$4^{-1}$</th>
<th>$m_x$</th>
<th>$m_+$</th>
<th>$m_y$</th>
<th>$m_-$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>$4^{-1}$</td>
<td>$m_x$</td>
<td>$m_+$</td>
<td>$m_y$</td>
<td>$m_-$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>$4^{-1}$</td>
<td>4</td>
<td>$m_y$</td>
<td>$m_-$</td>
<td>$m_x$</td>
</tr>
<tr>
<td>4</td>
<td>$4^{-1}$</td>
<td>2</td>
<td>1</td>
<td>$m_+$</td>
<td>$m_y$</td>
<td>$m_-$</td>
<td>$m_x$</td>
</tr>
<tr>
<td>$4^{-1}$</td>
<td>$4^{-1}$</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>$m_-$</td>
<td>$m_x$</td>
<td>$m_+$</td>
</tr>
</tbody>
</table>

$m_x, m_x m_y m_- m_+ 1 4^{-1} 2 4$
$m_+, m_+ m_- m_x m_y 4 1 4^{-1} 2$
$m_y, m_y m_x m_+ m_- 2 4 1 4^{-1}$
$m_- m_- m_+ m_y m_x 4^{-1} 2 4 1$
Problem 2.12 (additional)

(i) Consider the group of the equilateral triangle and determine its subgroups;

(ii) Distribute the subgroups into classes of conjugate subgroups;

(iii) Construct the maximal subgroup graph of $3m$

\[
\begin{array}{c|cccccc}
\times & 1 & 3^+ & 3^- & m_{xx} & m_x & m_y \\
\hline
1 & 1 & 3^+ & 3^- & m_{xx} & m_x & m_y \\
3^+ & 3^+ & 3^- & 1 & m_y & m_{xx} & m_x \\
3^- & 3^- & 1 & 3^+ & m_x & m_y & m_{xx} \\
m_{xx} & m_{xx} & m_x & m_y & 1 & 3^+ & 3^- \\
m_x & m_x & m_y & m_{xx} & 3^- & 1 & 3^+ \\
m_y & m_y & m_{xx} & m_x & 3^+ & 3^- & 1 \\
\end{array}
\]
Coset decomposition $G:H$

Group-subgroup pair $H < G$

left coset decomposition $G = H + g_2H + ... + g_mH$, $g_i \notin H$,
$m = \text{index of } H \text{ in } G$

right coset decomposition $G = H + Hg_2 + ... + Hg_m$, $g_i \notin H$
$m = \text{index of } H \text{ in } G$

Coset decomposition-properties

(i) $g_iH \cap g_jH = \{\emptyset\}$, if $g_i \notin g_jH$

(ii) $|g_iH| = |H|$

(iii) $g_iH = g_jH$, $g_i \in g_jH$
Coset decomposition $G:H$

Normal subgroups

Theorem of Lagrange

Group $G$ of order $|G|$

Subgroup $H < G$ of order $|H|$

then

$|H|$ is a divisor of $|G|$

and $[i] = |G:H|$

Corollary

The order $k$ of any element of $G$, $g^k = e$, is a divisor of $|G|$
Problem 2.13

Consider the subgroup \{e,2\} of 4mm, of index 4:

- Write down and compare the right and left coset decompositions of 4mm with respect to \{e,2\};

- Are the right and left coset decompositions of 4mm with respect to \{e,2\} equal or different? Can you comment why?

Problem 2.14

Demonstrate that H is always a normal subgroup if |G:H|=2.
Conjugate subgroups

Let $H_1 < G, H_2 < G$ then, $H_1 \sim H_2$, if $\exists g \in G: g^{-1}H_1g = H_2$

(i) Classes of conjugate subgroups: $L(H)$

(ii) If $H_1 \sim H_2$, then $H_1 \cong H_2$

(iii) $|L(H)|$ is a divisor of $|G|/|H|$

Normal subgroup

$H \triangleleft G$, if $g^{-1}Hg = H$, for $\forall g \in G$
Problem 2.11 (cont)  Distribute the subgroups of the group of the square into classes of conjugate subgroups

**Hint:** The stereographic projections could be rather helpful
Complete and contracted group-subgroup graphs

Complete graph of maximal subgroups

Contracted graph of maximal subgroups
Fig. 10.1.3.2. Maximal subgroups and minimal supergroups of the three-dimensional crystallographic point groups. Solid lines indicate maximal normal subgroups; double or triple solid lines mean that there are two or three maximal normal subgroups with the same symbol. Dashed lines refer to sets of maximal conjugate subgroups. The group orders are given on the left. Full Hermann–Mauguin symbols are used.
GENERAL AND SPECIAL
WYCKOFF POSITIONS
General and special Wyckoff positions

Site-symmetry group $S_o=\{W\}$ of a point $X_o$

$$WX_o = X_o$$

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>$x_0$</th>
<th>$x_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>d</td>
<td>e</td>
<td>f</td>
<td>$y_0$</td>
<td>$y_0$</td>
</tr>
<tr>
<td>g</td>
<td>h</td>
<td>i</td>
<td>$z_0$</td>
<td>$z_0$</td>
</tr>
</tbody>
</table>

General position $X_o$

$S=1=\{1\}$

Special position $X_o$

$S>1=\{1,...,\}$

Site-symmetry groups: oriented symbols
Point group $2 = \{1, 2_z\}$

Site-symmetry group $S_\circ = \{W\}$ of a point $X_\circ = (0,0,z)$

$S_\circ = 2$

$WX_\circ = X_\circ$

Example

\[
\begin{array}{ccc}
2 & b & 1 \\
1 & a & 2 \\
\end{array}
\]

$(x,y,z)$  
$(-x,-y,z)$

$(0,0,z)$
General and special Wyckoff positions

Point group $\text{mm2} = \{1, 2z, m_x, m_y\}$

Site-symmetry group $S_0 = \{W\}$ of a point $X_0 = (0, 0, 0)$

$S_0 = \text{mm2}$

$WX_0 = X_0$

Example:

2z:

\[
\begin{array}{ccc}
-1 & 0 & 0 \\
-1 & 0 & z \\
1 & z & z \\
\end{array}
\]

$m_y$:

\[
\begin{array}{ccc}
1 & 0 & 0 \\
-1 & 0 & z \\
1 & z & z \\
\end{array}
\]

4 d 1

$(x, y, z)$

$(-x, -y, z)$

$(x, -y, z)$

$(-x, y, z)$

2 c m.. $(0, y, z)$ $(0, -y, z)$

2 b .m. $(x, 0, z)$ $(-x, 0, z)$

1 a mm2 $(0, 0, z)$

miércoles, 25 de septiembre de 13
Consider the symmetry group of the square $4\text{mm}$ and the point group $4\text{22}$ that is isomorphic to it.

Determine the general and special Wyckoff positions of the two groups.

*Hint:* The stereographic projections could be rather helpful.
**EXAMPLE**

Wyckoff positions splitting schemes

Group-subgroup pair \( mm2 > 2, [i]=2 \)

\[ mm2 \]

\[ 2 \]

**4 d 1**

\((x,y,z)\)

\((-x,-y,z)\)

\((x,-y,z)\)

\((-x,y,z)\)

\[ x,y,z = x_1,y_1,z_1 \]

\[ -x,-y,z = -x_1,-y_1,z_1 \]

\[ x,-y,z = x_2,y_2,z_2 \]

\[ -x,y,z = -x_2,-y_2,z_2 \]
Consider the general and special Wyckoff positions of the symmetry group of the square $4mm$ and those of its subgroup $mm2$ of index 2.

Determine the splitting schemes of the general and special Wyckoff positions for $4mm > mm2$.

*Hint:* The stereographic projections could be rather helpful.
NORMALIZERS
Normalizer of $H$ in $G$

Normal subgroup

$H \triangleleft G$, if $g^{-1}Hg = H$, for $\forall g \in G$

Normalizer of $H$ in $G$, $H \triangleleft G$

$\text{NG}(H) = \{g \in G, \text{ if } g^{-1}Hg = H\}$

$G \geq \text{NG}(H) \geq H$

What is the normalizer $\text{NG}(H)$ if $H \triangleleft G$?

Number of subgroups $H_i \triangleleft G$ in a conjugate class

$n = [G:\text{NG}(H)]$
Problem 2.18

Consider the group $4mm$ and its subgroups of index 4. Determine their normalizers in $4mm$. Distribute the subgroups into conjugacy classes with the help of their normalizers in $4mm$.

**Hint:** The stereographic projections could be rather helpful.
ADDITIONAL
GROUP-SUPERGROUP
RELATIONS
Supergroups: Some basic results (summary)

Supergroup $G > H$

$H = \{ e, h_1, h_2, ..., h_k \} \subset G$

Proper supergroups $G > H$, and trivial supergroup: $H$

Index of the group $H$ in supergroup $G$: $[i] = |G|/|H|$ (order of $G$)/(order of $H$)

Minimal supergroups $G$ of $H$

NO subgroup $Z$ exists such that: $H < Z < G$
The Supergroup Problem

Given a group-subgroup pair $G > H$ of index $[i]$

Determine: all $G_k > H$ of index $[i]$, $G_i \cong G$

all $G_k > H$ contain $H$ as subgroup

$G_k = H + g_2 H + ... + g_{ik} H$
Example: Supergroup problem

Group-subgroup pair
422 > 222

Supergroups 422 of the group 222

How many are the subgroups 222 of 422?

How many are the supergroups 422 of 222?
Example: Supergroup problem

Group-subgroup pair

422 > 222

Supergroups 422 of the group 222

\[ 4z_{22} = 2z_2 \times 2y + 4z(2z_2 \times 2y) \]
\[ 4z_{22} = 2z_2 + 2z + 4z(2z_2 + 2z) \]

\[ 4y_{22} = 222 + 4y_{222} \]
\[ 4x_{22} = 222 + 4x_{222} \]
GENERATION OF CRYSTALLOGRAPHIC POINT GROUPS
Generation of point groups

Crystallographic groups are **solvable** groups

**Composition series:** $1 \triangleleft \mathbb{Z}_2 \triangleleft \mathbb{Z}_3 \triangleleft \ldots \triangleleft G$

index 2 or 3

**Set of generators** of a group is a set of group elements such that each element of the group can be obtained as an ordered product of the generators

$$W = (g_h)^{k_h} \ast (g_{h-1})^{k_{h-1}} \ast \ldots \ast (g_2)^{k_2} \ast g_1$$

$g_1$ - identity

$g_2, g_3, \ldots$ - generate the rest of elements
Example Generation of the group of the square

**Composition series:** \( I \vartriangleleft 2 \vartriangleleft 4 \vartriangleleft 4mm \)

**Step 1:**
\( I = \{1\} \)

**Step 2:**
\( 2 = \{1\} + 2z \{1\} \)

**Step 3:**
\( 4 = \{1,2\} + 4z \{1,2\} \)

**Step 4:**
\( 4mm = 4 + m_x 4 \)

**Multiplication table of 4mm**
Generation of sub-cubic point groups

\[
\begin{align*}
\bar{m}3m & \quad \leftrightarrow \quad 432 \quad (43\overline{m}) \\
\bar{m}3 & \quad \leftrightarrow \quad 23 \\
\bar{m}m & \quad \leftrightarrow \quad 222 \quad (mm\overline{2}) \\
m\bar{m} & \quad \leftrightarrow \quad 4/mmm \\
m & \quad \leftrightarrow \quad 4/m \\
\end{align*}
\]
### Composition series of cubic point groups and their subgroups

<table>
<thead>
<tr>
<th>HM Symbol</th>
<th>SchoeSy</th>
<th>generators</th>
<th>compos. series</th>
</tr>
</thead>
<tbody>
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<td>$T$</td>
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<td>$m\bar{3}m \triangleright 432 \triangleright \ldots$</td>
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</table>
Generation of sub-hexagonal point groups
Composition series of hexagonal point groups and their subgroups

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<th>HM Symbol</th>
<th>SchoeSy</th>
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<th>compos. series</th>
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</table>
Problem 2.15

Generate the symmetry operations of the group $4/mmm$ following its composition series.

Generate the symmetry operations of the group $\overline{3}m$ following its composition series.