The Dirac Delta function

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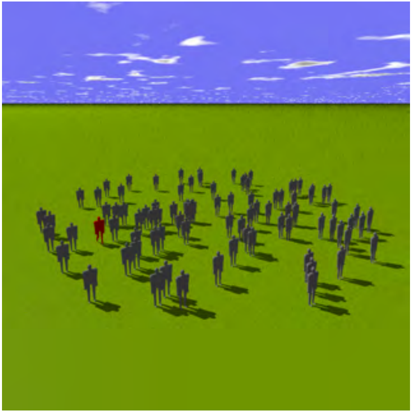
Outline

The Kronecker Delta



Suppose we have a sequence of values $\{a_1, a_2, \ldots\}$ and we wish to select algebraically a particular value labeled by its index i

The Kronecker Delta

Definition (Kronecker delta)

$$ {}^K\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} $$

Picking one member of a set algebraically

$$ \sum_{j=1} a_j \, {}^K\delta_{ij} = a_i $$

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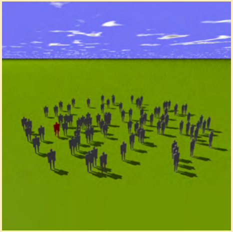

$$\sum_{j=1} a_j \, {}^K\delta_{ij} = a_i$$

Properties of the Kronecker Delta

- $\sum_j {}^K\delta_{ij} = 1$: Normalization condition.
- ${}^K\delta_{ij} = {}^K\delta_{ji}$: Symmetry property.

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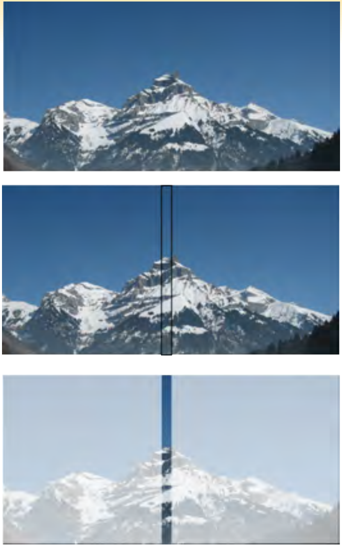
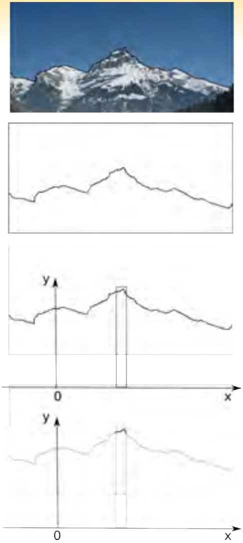
Spotting a point in the mountain profile



We want to pick up just a narrow window of the whole view

How to deal with continuous functions ?



We want to do the same with a continuous function.

Defining the Dirac Delta function

Consider a function $f(x)$ continuous in the interval (a, b) and suppose we want to pick up algebraically the value of $f(x)$ at a particular point labeled by x_0.
In analogy with the Kronecker delta let us define a selector function ${}^D\delta(x)$ with the following two properties:

- $\int_a^b f(x)\, {}^D\delta(x - x_0)dx = f(x_0)$: Selector or sifting property
- $\int_a^b {}^D\delta(x - x_0)dx = 1$: Normalization condition

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Defining the Dirac Delta function

To be more general consider $f(x)$ to be continuous in the interval (a, b) except in a finite number of points where finite discontinuities occurs, then the Dirac Delta can be defined as

Definition (Dirac delta function)

$$\int_a^b f(x)\delta(x - x_0)dx = \begin{cases} \frac{1}{2}[f(x_0^-) + f(x_0^+)] & x_0 \in (a, b) \\ \frac{1}{2}f(x_0^+) & x_0 = a \\ \frac{1}{2}f(x_0^-) & x_0 = b \\ 0 & x_0 \notin (a, b) \end{cases}$$

A bit of history

Siméon Denis Poisson (1781-1840)

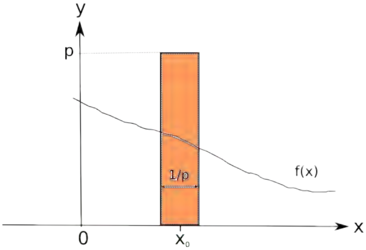
In 1815 Poisson already for sees the $\delta(x - x_0)$ as a selector function using for this purpose Lorentzian functions. Cauchy (1823) also made use of selector function in much the same way as Poisson and Fourier gave a series representation on the delta function(More on this later).

Paul Adrien Maurice Dirac (1902-1984)

Dirac "rediscovered" the delta function that now bears his name in analogy for the continuous case with the Kronecker delta in his seminal works on quantum mechanics.

What does it look like ?

$$\delta_p(x - x_0) = \begin{cases} p & x \in (x_0 - 1/2p, x_0 + 1/2p) \\ 0 & x \notin (x_0 - 1/2p, x_0 + 1/2p) \end{cases}$$



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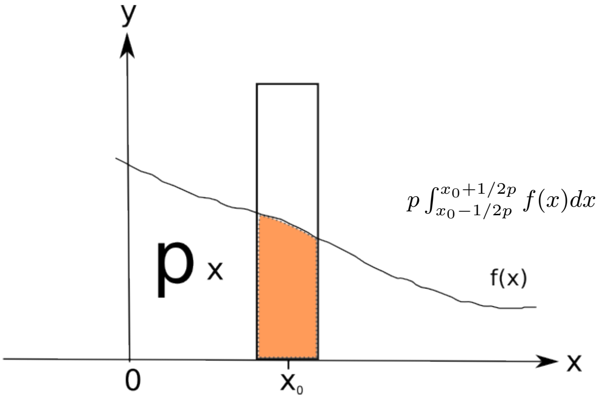

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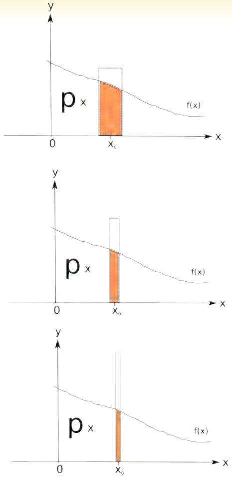
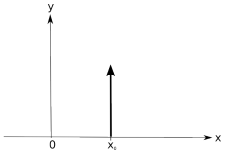


$$\int_{-\infty}^{\infty} \delta_p(x)\, dx = \int_{-1/2p}^{1/2p} p\, dx = 1$$

Definition (Dirac delta function)

$$\delta(x - x_0)dx = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

What does it look like ?



Definition (Dirac delta function)

$$\delta(x - x_0) = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

- The above expression is just "formal", the $\delta(x)$ must be always understood in the context of its selector property i.e. within the integral
- $\delta(x)$ is defined more rigorously in terms of a distribution or a functional (generalized function)

Dirac delta function as the limit of a family of functions

The Dirac delta function can be pictured as the limit in a sequence of functions δ_p which must comply with two conditions:

- $\lim_{p\to\infty} \int_{-\infty}^{\infty} \delta_p(x)dx = 1$: Normalization condition
- $\lim_{p\to\infty} \frac{\delta_p(x\neq 0)}{\lim_{x\to 0} \delta_p(x)} = 0$ Singularity condition.

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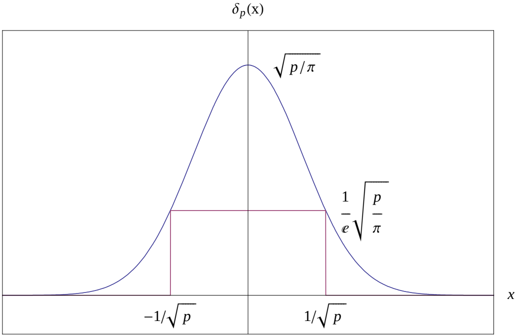
… as the limit of Gaussian functions

$\delta_p(x)$ Gaussian family

$$\delta_p(x) = \sqrt{\frac{p}{\pi}} \exp\left(-px^2\right)$$



... as the limit of Gaussian functions

Normalization condition

$$\int_{-\infty}^{\infty} \delta_p(x)dx = \sqrt{\frac{p}{\pi}}\int_{-\infty}^{\infty} \exp{(-px^2)}dx = \sqrt{\frac{1}{\pi}}\int_{-\infty}^{\infty} \exp{(-px^2)}d(\sqrt{p}x) =$$
$$\sqrt{\frac{1}{\pi}}\int_{-\infty}^{\infty} \exp{(-t^2)}dt = 2\sqrt{\frac{1}{\pi}}\int_{0}^{\infty} \exp{(-t^2)}dt$$

$I = \int_0^\infty e^{-t^2}dt$

$I^2 = \int_0^\infty e^{-y^2}dy \int_0^\infty e^{-z^2}dz = \int\int_0^\infty \exp{(y^2+z^2)}dydz$

Normalization condition

$$\left\{ \begin{array}{c} r^2 = y^2 + z^2 \\ y = r\cos\phi \\ z = r\sin\phi \end{array} \right\}$$

$$I^2 = \int_0^{\pi/2} d\phi \int_0^{\infty} e^{-r^2} r dr = \frac{\pi}{4} \int_0^{\infty} e^{-s} ds = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$

$$\int_{-\infty}^{\infty} \delta_p(x) dx = 1$$

... as the limit of Gaussian functions

Singularity condition

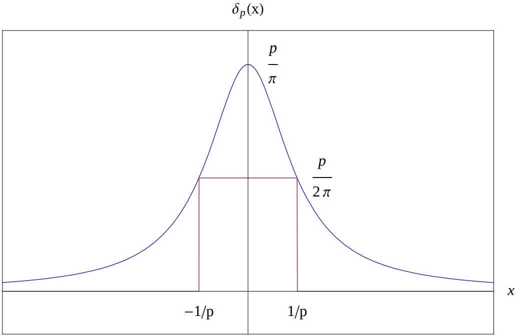
$$\lim_{p\to\infty} \frac{\delta_p(x \neq 0)}{\lim_{x\to 0} \delta_p(x)} =$$

$$\lim_{p\to\infty} \frac{\sqrt{\frac{p}{\pi}} e^{-px^2}}{\sqrt{\frac{p}{\pi}}} = 0$$

... as the limit of Lorentzian functions

$\delta_p(x)$ Lorentzian family

$$\delta_p(x) = \frac{1}{p}\frac{p}{1+p^2x^2}$$

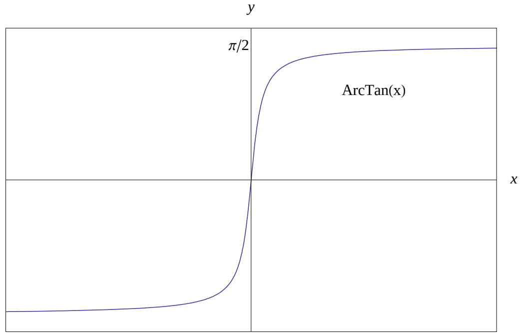


... as the limit of Lorentzian functions

Normalization condition

$$\int_{-\infty}^{\infty} \delta_p(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{pdx}{1+p^2x^2} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} =$$

$$\frac{1}{\pi}\lim_{k\to\infty} \arctan t|_{t=-k}^{t=k} = \frac{2}{\pi}\lim_{k\to\infty} \arctan k = 1$$



... as the limit of Lorentzian functions

Singularity condition

$$\lim_{p\to\infty} \frac{\delta_p(x \neq 0)}{\lim_{x\to 0} \delta_p(x)} =$$

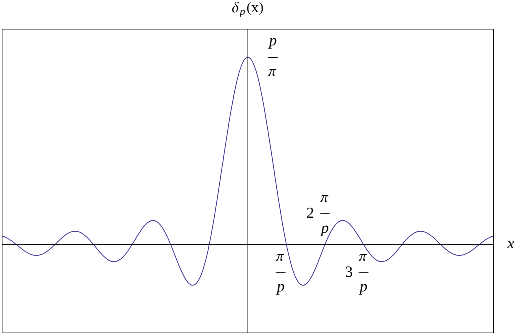
$$\frac{1}{\pi} \lim_{p\to\infty} \frac{\frac{p}{1+p^2x^2}}{p} =$$

$$\frac{1}{\pi} \lim_{p\to\infty} \frac{1}{1+p^2x^2} = 0$$

... as the limit of Sinc functions

$\delta_p(x)$ Sinc family

$$\delta_p(x) = \frac{p}{\pi}\frac{\sin px}{px}$$



... as the limit of Sinc functions

Normalization condition

$$\int_{-\infty}^{\infty} \delta_p(x)dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin{(px)}dx}{x} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin z}{z}dz = \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin z}{z}dz = \frac{2}{\pi}\frac{\pi}{2} = 1$$

… as the limit of Sinc functions

Singularity condition

$$\lim_{p\to\infty} \frac{\delta_p(x \neq 0)}{\lim_{x\to 0} \delta_p(x)} =$$

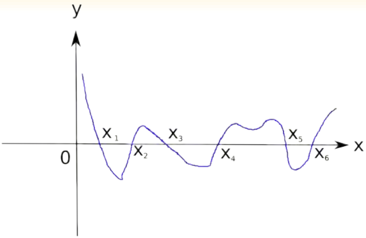
$$\lim_{p\to\infty} \frac{\frac{1}{\pi}\frac{\sin{(px)}}{x}}{\frac{p}{\pi}} = \lim_{p\to\infty} \frac{\sin{(px)}}{px} = 0$$

$\delta_p(x)$ alternative definition of the Sinc family

$\delta_p(x) = \frac{1}{2\pi}\int_{-p}^{p} e^{\pm itx} dt$

$\delta_p(x) = \frac{1}{\pi}\int_{0}^{p} \cos{(tx)} dt$

Properties of the Dirac delta function



Let us denote by x_n the roots of the equation $f(x) = 0$ and suppose that $f^{'}(x_n) \neq 0$ then

Composition of functions

$$\delta(f(x)) = \sum_n \frac{\delta(x - x_n)}{|f^{'}(x_n)|}$$

Important consequences of the composition property are

- $\delta(-x) = \delta(x)$ (symmetry property).
- $\delta(ax) = \frac{\delta(x)}{|a|}$ (scaling property).
- $\delta(ax - x_0) = \frac{\delta(x - \frac{x_0}{a})}{|a|}$ (a more general formulation of the scaling property).

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Convolution

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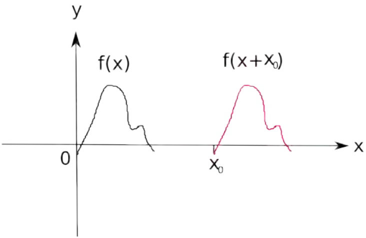
$$f(x)\otimes\delta(x+x_0) = \int_{-\infty}^{\infty} f(\varsigma)\delta(\varsigma-(x+x_0))d\varsigma = f(x+x_0)$$



The effect of convolving with the position-shifted Dirac delta is to shift $f(t)$ by the same amount.

- $\int_{-\infty}^{\infty} \delta(\zeta - x)\delta(x-\eta)dx = \delta(\zeta-\eta))$

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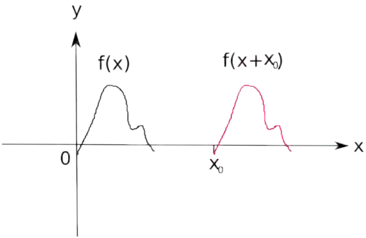

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Heaviside

Definition (Step function)

$\Theta(x) = \frac{1}{2}(1 + \frac{x}{|x|})$

Definition (Step function)

$$\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\frac{d\Theta}{dx} = \frac{1}{2}\frac{d}{dx}\frac{x}{|x|} = \frac{1}{2}\lim_{p\to\infty}\frac{d}{dx}\frac{2}{\pi}\arctan(px) =$$
$$\frac{1}{\pi}\lim_{p\to\infty}\frac{p}{1+p^2x^2} = \delta(x)$$

Heaviside

$\delta(x) = \frac{d\Theta(x)}{dx}$

Heaviside

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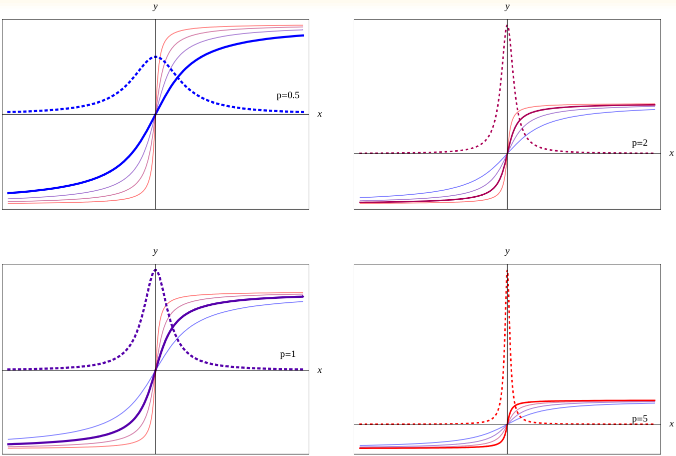

Fourier transform of the Dirac delta function

Fourier transform

$\Gamma[\delta(x)] = \widehat{\delta}(x^*) \equiv \int_{-\infty}^{\infty} \delta(x) \exp(-2\pi i x^* x) dx = 1$

This property allow us to state yet another definition of the Dirac delta as the inverse Fourier transform of $f(x) = 1$

Definition (Dirac delta function)

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Definition (Dirac delta function)

$$\delta(x) = \int_{-\infty}^{\infty} \exp(2\pi i x^* x) dx^*$$

Dirac delta function obtained from a complete set of orthonormal functions

Let the set of functions $\{\psi_n\}$ be a complete system of orthonormal functions in the interval (a, b) and let x and x_0 be inner points of that interval. Then

Theorem (Orthonormal functions)

$\sum_n \psi_n^*(x)\psi_n(x_0) = \delta(x - x_0)$

To proof the theorem we shall demonstrate that the left hand side has the sifting property of the Dirac distribution

$I = \int_a^b f(x) \sum_n \psi_n^*(x)\psi_n(x_0)dx = f(x_0)$

$f(x) = \sum_m c_m\psi_m(x) \qquad c_m = \int_a^b f(x)\psi_m^*(x)dx$

Dirac delta function obtained from a complete set of orthonormal functions

$$I = \int_a^b \sum_m c_m \psi_m(x) \sum_n \psi_n^*(x) \psi_n(x_0) dx =$$

$$\sum_m c_m \sum_n \psi_n(x_0) \int_a^b \psi_n^*(x) \psi_m(x) dx$$

$\int_a^b \psi_n^*(x) \psi_m(x) dx = {}^K\delta_{mn}$

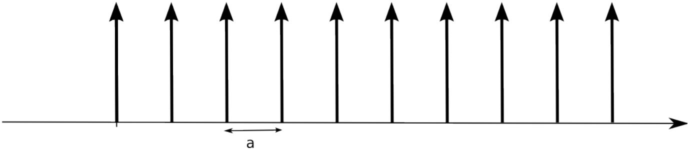
$$I = \sum_n c_m \sum_n \psi_n(x_0) \, {}^K\delta_{mn} =$$

$$\sum_n c_m \psi_m(x_0) = f(x_0)$$

Dirac comb

Definition (Dirac comb)

$\sum_{m=-\infty}^{\infty} \delta(x - ma)$



Dirac comb

The set $\{\frac{1}{\sqrt{(|a|)}}\exp\left(2\pi inx/a\right)\}$ forms a complete set of orthonormal functions, then

$$\delta(x) = \frac{1}{|a|}\sum_{n=-\infty}^{\infty}\exp\left(2\pi inx/a\right)$$

each summand in the LHS in the above expression is periodic with period $|a|$ therefore the whole sum is periodic with the same period and

Dirac comb

$$\sum_{m=-\infty}^{\infty}\delta(x-ma) = \frac{1}{|a|}\sum_{n=-\infty}^{\infty}\exp\left(2\pi inx/a\right)$$

Fourier transform of a Dirac comb

$$\int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(x - ma) \exp(-2\pi i x^* x) dx =$$

$$\sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - ma) \exp(-2\pi i x^* x) dx = \sum_{m=-\infty}^{\infty} \exp(2\pi i x^* ma)$$

Theorem (Fourier transform of a Dirac comb)

$\mathit{\Gamma}[\sum_{m=-\infty}^{\infty} \delta(x - ma)] = \frac{1}{|a|} \sum_{h=-\infty}^{\infty} \delta(x^* - h/a) = |a^*| \sum_{h=-\infty}^{\infty} \delta(x^* - ha^*)$

The Fourier transform of a Dirac comb is a Dirac comb

Dirac delta in higher dimensional space

Dirac delta in higher dimensions. Cartesian coordinates

$\int \cdots \int f(\vec{x})\delta(\vec{x}-\vec{x_0})d^N x = f(\vec{x_0})$

which comes from

$\int \cdots \int f(x_1, x_1, \ldots x_N)\delta(x_1 - x_{01})\delta(x_2 - x_{02})\ldots\delta(x_N - x_{0N})dx_1 dx_2 \ldots dx_N = f(x_{01}, x_{02}\vec{,} \ldots x_{0N})$

Definition (Dirac delta in higher dimensions. Cartesian coordinates)

$\delta(\vec{x}-\vec{x_0}) = \delta(x_1 - x_{01})\delta(x_2 - x_{02})\ldots\delta(x_N - x_{0N}) = \prod_{s=1}^{N}\delta(x_s - x_{0s})$

General coordinates

$\{x_i\}$: Cartesian coordinates
$\{y_i\}$: General coordinates

$$
\begin{aligned}
x_1 &= x_1(y_1, \dots y_N) \\
x_2 &= x_2(y_1, \dots y_N) \\
&\dots \\
x_N &= x_N(y_1, \dots y_N)
\end{aligned}
$$

$$
J(y_1, y_2, \dots, y_N) = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \dots & \frac{\partial x_1}{\partial y_N} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \dots & \frac{\partial x_2}{\partial y_N} \\ \dots & \dots & \dots & \dots \\ \frac{\partial x_N}{\partial y_1} & \frac{\partial x_N}{\partial y_2} & \dots & \frac{\partial x_N}{\partial y_N} \end{vmatrix}
$$

Definition (Dirac delta in higher dimensions. General coordinates)

$$\delta(\vec{x} - \vec{x_0}) = \frac{1}{|J(y_1, y_2, \dots, y_N)|} \delta(\vec{y} - \vec{y_0})$$

Oblique coordinates

$$x_1 = a_{11}y_1 + a_{12}y_2 \dots a_{1N}y_N$$
$$x_2 = a_{11}y_1 + a_{22}y_2 \dots a_{2N}y_N$$
$$\vdots$$
$$x_N = a_{N1}y_1 + a_{N2}y_2 \dots a_{NN}y_N$$

$$J(y_1, y_2, \dots, y_N) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{vmatrix}$$

$$\vec{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ \dots & \dots & \dots & \dots \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{pmatrix} \vec{y}$$

Definition (Dirac delta in higher dimensions. Oblique coordinates)

$$\delta(\vec{x} - \vec{x_0}) = \frac{1}{|\,||A||\,|}\delta(\vec{y} - \vec{y_0}) = \frac{1}{\sqrt{|\,||G||\,|}}\delta(\vec{y} - \vec{y_0})$$

Recap: Definitions

$$\delta(x - x_0)dx = \begin{cases} \infty & x = x_0 \\ 0 & x \neq x_0 \end{cases}$$

$$\delta(x) = \frac{d}{dx}\left(\frac{1}{2}(1 + \frac{x}{|x|})\right)$$

$$\delta(x) = \lim_{p\to\infty} \sqrt{\frac{p}{\pi}} \exp\left(-px^2\right)$$

$$\delta(x) = \lim_{p\to\infty} \frac{1}{p}\frac{p}{1+p^2x^2}$$

$$\delta(x) = \lim_{p\to\infty} \frac{p}{\pi}\frac{\sin px}{px}$$

$$\delta(x) = \int_{-\infty}^{\infty} \exp\left(2\pi i x^* x\right) dx^*$$

$$\delta(x - x_0) = \sum_n \psi_n^*(x)\psi_n(x_0)$$

where $\{\psi_n^*(x)\}$ is a complete set of orthornormal functions.

Recap: Properties

$$\int_a^b f(x)\delta(x-x_0)dx = \begin{cases} \frac{1}{2}[f(x_0^-)+f(x_0^+)] & x_0 \in (a,b) \\ \frac{1}{2}f(x_0^+) & x_0 = a \\ \frac{1}{2}f(x_0^-) & x_0 = b \\ 0 & x_0 \notin (a,b) \end{cases}$$

$$\delta(f(x)) = \sum_n \frac{\delta(x-x_n)}{|f'(x_n|}$$

$$\int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_n \frac{g(x_n)}{|f'(x_n)|}$$

$$\delta(-x) = \delta(x)$$

$$\delta(ax-x_0) = \frac{\delta(x-\frac{x_0}{a})}{|a|}$$

$$\delta(x^2-a^2) = \frac{\delta(x-a)+\delta(x+a)}{2|a|}$$

$$f(x)\otimes\delta(x+x_0) = f(x+x_0)$$

$$\mathit{\Gamma}[\sum_{m=-\infty}^{\infty}\delta(x-ma)] = |a^*|\sum_{h=-\infty}^{\infty}\delta(x^*-ha^*)$$

$$\mathit{\Gamma}[\delta(x)] = \widehat{\delta}(x^*) = 1$$

$$\sum_{m=-\infty}^{\infty}\delta(x-ma) = \frac{1}{|a|}\sum_{n=-\infty}^{\infty}\exp\left(2\pi inx/a\right)$$

Recap: Higher Dimensions

Cartesian coordinates:

$$\int \cdots \int f(\vec{x})\delta(\vec{x}-\vec{x_0})d^N x = f(\vec{x_0})$$

$$\delta(\vec{x}-\vec{x_0}) = \delta(x_1 - x_{01})\delta(x_2 - x_{02})\ldots\delta(x_N - x_{0N}) = \prod_{s=1}^{N}\delta(x_s - x_{0s})$$

General Coordinates $\{y_i\}$:

$$\delta(\vec{x}-\vec{x_0}) = \frac{1}{|J(y_1, y_2, \ldots, y_N)|}\delta(\vec{y}-\vec{y_0})$$

Oblique Coordinates $\vec{x} = A \cdot \vec{y}$:

$$\delta(\vec{x}-\vec{x_0}) = \frac{1}{|\,||A||\,|}\delta(\vec{y}-\vec{y_0}) = \frac{1}{\sqrt{|\,||G||\,|}}\delta(\vec{y}-\vec{y_0})$$

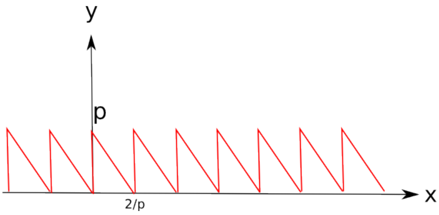
Exercises

Prove:

- $\delta(f(x)) = \sum_n \frac{\delta(x-x_n)}{|f'(x_n|}$ (Hint: develop $f(x)$ in Taylor series around x_n and prove the sifting property with $\delta(f(x))$)
- $\sum_{n=-\infty}^{\infty} f(na) = |x^*| \sum_{m=-\infty}^{\infty} \widehat{f}(mx^*)$ (Poisson summation formula)
- $\text{lím}_{p\to\infty} \frac{\sin\left((2p+1)\pi\frac{x-x_0}{a}\right)}{\sin\left(\pi\frac{x-x0}{a}\right)} = |a| \sum_{m=-\infty}^{\infty} \delta(x - x_0 - ma)$

Exercises

Prove that on the limit $p \to \infty$ the sawtooth funtion tends to the Dirac comb



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All about the Dirac Delta Function(?)